1 INTRODUCTION

1.1 Notations

$\forall$	For all
$\perp$	Orthogonal
$\neq$	Not Equal
$\subset$	Is a proper subset of
$\subseteq$	Subset (may be the same)
$\in$	Belongs to
$\bar{x}$	complex conjugate of x
$\exists$	There exists
$\exists!$	There exists a unique
$\ni$	Such that
$s.t.$	Such that
$\cup$	Union
$\cap$	Intersection
$\Rightarrow$	Implies
$\Leftrightarrow$	If and only if
iff	If and only if
$\oplus$	Direct sum of subspaces
$\overset{\perp}{\oplus}$	Direct sum of orthogonal subspaces
$\overset{\triangle}{=}$	Definition
$\mathcal{R}$	Real line (number)
$\mathcal{R}^+$	Positive real line (numbers)
$\mathcal{C}$	Complex scalars
$\mathcal{R}^n$	Space of n-vectors, each entry in $\mathcal{R}$
$\mathcal{C}^n$	Space of n-vectors, each entry in $\mathcal{C}$

1.2 Brief Introduction

This hand-out is an attempt to cover the material needed in the class. It is meant to be used as your primary source of information. Together with the references, it should give you a good start to your pursuit of research topics in the controls area. As far as overview is concerned (!) we will be following a rather strange path that connects the old fashioned SISO (single input single output) control techniques to the modern control techniques you have seen by now.

Classical control techniques were immensely powerful. To this date, it is unlikely (but not impossible!) that a new graduate student with lots of new 'modern' tricks can do a better job designing a control law for a SISO system (of low to moderate order), than an engineer who has a few years of experience with Bode and Nyquist plots, Bode integrals, etc. Admittedly, new techniques have given a few new wrinkles in the analysis part, but not much can be said about the synthesis part.

The problem encountered since the 60's are: Large order systems (which make transfer functions hard to deal with), MIMO settings and the notation of explicit parameter variation (or structured uncertainties). All of these could be handled (to varying degrees of success) with modern (a.k.a., state space based) methods. This was the main focus of work in the 60's and early 70's.

In the late 70's, people noticed that the 'down side' of using these new methods was loss or degradation of some of the classical (and critical) properties of control systems. Since then, there has been a great deal of attention paid to bridging these two approaches in an attempt to address all of the new and old concerns.

It all started in the LQR and LQG setting (later called H_2 for marketing purposes) which eventually lead to the world famous H_∞ framework. More recently, the H_∞ framework has been used to develop results for a variety of more difficult problems; such as time varying problems, multi-objective problems, hybrid systems etc. The real breakthrough, though, is recent development of powerful numerical tools for finding solution to convex searches (e.g., via the LMI toolbox of MATLAB!!). This has allowed a large number of problems be solved numerically, even though closed form solutions are not apparent. In order to appreciate this progress however, a load of basic results from linear algebra and control theory is needed. We start with the basic stuff for quite a few weeks, so that we can spend the last 3 weeks or so on the new results.

Please read carefully and report any typos (for which I accept no responsibility!) to the authorities.

2 PRELIMINARIES

2.1 Norms, inner products, etc.

Definition 2.1. *A real scalar valued function, $\|.\|$, is a norm if it satisfies three properties:*

- $\|x\| \geq 0, \quad and \quad \|x\| = 0$ *iff* $x = 0$
- $\|\alpha x\| = |\alpha| \, \|x\|, \quad \forall \; \alpha \in \mathcal{C}$
- $\|x + y\| \leq \|x\| + \|y\| \quad$ *(Triangle inequality)*

Definition 2.2. *A complex valued function, $\langle ., . \rangle$, is an inner product if it satisfies the following basic properties:*

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle \alpha x, (\beta_1 y_1 + \beta_2 y_2) \rangle = \overline{\alpha} \, \beta_1 \langle x, y_1 \rangle + \overline{\alpha} \, \beta_2 \langle x, y_2 \rangle \quad \forall \quad \alpha \, , \, \beta_1 \, , \, \beta_2 \quad \in \mathcal{C}$
- $\langle x, x \rangle \geq 0, \quad and \; \langle x, x \rangle = 0$ *iff* $x = 0$

Remark 2.3. *A very common norm is the one based on the inner product; i.e.*

$$\|x\|^2 = \langle x, x \rangle \tag{2.1}$$

Definition 2.4. *Two vectors are orthogonal if their inner product is zero; i.e.,*

$$\langle x, y \rangle = 0 \Leftrightarrow x \perp y.$$

Furthermore, two subspaces are orthogonal to each other if any vector from one is orthogonal to any vector from the other; i.e.,

$$\mathcal{X} \perp \mathcal{Y} \Leftrightarrow \langle x, y \rangle = 0 \quad \forall \; x \; \in \mathcal{X} \, and \, \forall \quad y \; \in \; \mathcal{Y}.$$

Example 2.5. *For vectors in $\mathcal{C}^n$, the following are all norms:*

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2 \,)^{\frac{1}{2}}$$

$$\|x\|_\infty = \max_i \{|x_i|\}$$

where x_i is the i^{th} entry of vector x.

Example 2.6. *Consider scalar functions of time defined for $t \in [0, T]$. We can define the following norms*

$$\|x(.)\|_1 = \int_0^T |x(t)|\, dt$$

$$\|x(.)\|_2 = \left(\int_0^T |x(t)|^2\, dt \right)^{\frac{1}{2}}$$

$$\|x(.)\|_\infty = \max_{T \geq t \geq 0} \{|x(t)|\}$$

Example 2.7. *For vectors in $\mathcal{C}^n$, or $\mathcal{R}^n$, the following are all proper inner products*

$$\langle x, y \rangle = \overline{x}^T y$$

$$\langle x, y \rangle_M = \overline{x}^T M y, \quad \textit{for some } M > 0 \ (\textit{positive definite})$$

Example 2.8. *Consider scalar and continuous functions $x(t)$ and $y(t)$ defined for $t \in [0, T]$. We can define the following inner product*

$$\langle x, y \rangle_{L_2} = \int_0^T \overline{x(t)}\, y(t)\, dt$$

and, indeed, $x(t)$ and $y(t)$ can be n-vector as well!

2.2 Linear Operators

Consider two vector spaces $\mathcal{X}$ and $\mathcal{Y}$ (e.g., $\mathcal{R}^n$ and $\mathcal{R}^m$, but it could be more complicated and general). A function T that sends every vector $x \in \mathcal{X}$ to a vector $y = Tx \in \mathcal{Y}$ is called a linear operator (or linear transformation) on - or from - $\mathcal{X}$ to $\mathcal{Y}$ if it preserves linear relations; that is if

$$T(a_1x_1 + a_2x_2) = a_1Tx_1 + a_2Tx_2$$

for all x_1, and x_2 in $\mathcal{X}$, as well as all scalars a_1 and a_2. We often use the notation $T : \mathcal{X} \to \mathcal{Y}$. Matrices, derivatives, integrals, are examples of linear operators (what are $\mathcal{X}$, $\mathcal{Y}$, x, etc in each case?).

So such a T can be considered an operator. It can also be considered an element (vector) of a large collection of operators that share the main properties (i.e., linear operators from $\mathcal{X}$ to $\mathcal{Y}$). The big set or collection of these operators actually is a linear vector space itself (why?), often denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. So T can be considers a vector in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ as well, for which a vector norm can be obtained as well!

Example 2.9. *Every matrix is a linear operator (work all details)*

Example 2.10. *The operation $\frac{d}{dt}$ is a linear operator ($y = \dot{x} = \mathcal{L}x$). Note that clearly we have x from the space of continuous function while y may not be. Similarly, simple integrals are linear operators.*

Since linear operators are elements of sets (i.e., vectors in subspaces), one can define vector norms (or metric) for them. As an example, consider matrices that are $m \times n$: a common vector norm for this matrix is

$$\|A\|_F = (\sum_i \sum_j |a_{i,j}|^2)^{.5}$$

where $a_{i,j}$ is the (i,j) element of the matrix A. This norm, often called the Frobenius norm, may indicate the 'size' of a given matrix compared to other matrices.

A more interesting, and useful, way to 'size' a linear operator is by examining what it does to the vectors it operates on. We define the operator norm for the space of linear operators (i.e, $\mathcal{L}$).

Definition 2.11. *The operator norm, or the induced norm, of an operator A is the supremum of the ratio of $\|Ax\|$ to $\|x\|$, over all non zero x; i.e.*

$$\|A\|_i = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

where the subscript i is often used to emphasize the fact that the induced norm is used. When clear from the context, this subscript is dropped.

Remark 2.12. *The norms in the fraction are vectors norm, from potentially different spaces. While they are often the same; e.g., both are 2-norms, they could be mixed. A common example is energy to peak norm; i.e.,* $\frac{\|Ax\|_\infty}{\|x\|_2}$ *which measures the worst peak per unit of input energy.*

Remark 2.13. *The form used in* definition 2.11 *also applies to nonlinear operators and is used in many nonlinear analysis problems. For the linear case, we can also use* $\sup_{\|x\|=1} \|Ax\|$.

Remark 2.14. *'Sup' is the lowest upper bound. When it is achieved, it is 'max' (which is 99% of problems!). To calculate the induced norm, a three step process is followed: (1) Come up with an estimate (guess), (2) show it is a legitimate upper bound (i.e., bigger that the ratio for all possible* x*), and (3) show that with a clever choice of* x*, one can either achieve it or get arbitrarily close to it.*

Example 2.15. *For* $m \times n$ *matrices, we have*

$$\sup_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_j \sum_{i=1}^{m} |a_{i,j}|$$

$$\sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = [\, \lambda_{max}(\bar{A}^T A)\,]^{\frac{1}{2}}$$

$$\sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_i \sum_{j=1}^{n} |a_{i,j}|$$

Before we leave this subsection, it is useful to review the concept of adjoint operator.

Definition 2.16. *Consider the linear operator* A *from* $\mathcal{X}$ *to* $\mathcal{Y}$*, with inner products* $\langle .,. \rangle_x$ *and* $\langle .,. \rangle_y$ *defined on* $\mathcal{X}$ *and* $\mathcal{Y}$*, respectively. Then the adjoint operator* A^* *is defined by the operator that satisfies the following*

$$\langle Ax, y\rangle_y = \langle x, A^* y\rangle_x \;\forall\; x \in \mathcal{X}, \quad y \in \mathcal{Y}.$$

Example 2.17. *The simplest case to consider would be* A *in the space of real matrices of dimension* $m \times n$ *with basic* l_2 *inner product on both* $\mathcal{R}^n$ *and* $\mathcal{R}^m$*. It that case,* $A^* = A^T$*. What if the inner product was* $\langle x, y\rangle = x^T M y$*, for some* $M > 0$*? (show that* $A^* = M^{-1} A^T M$*).*

2.3 Orthogonal Projection

Let v_1, v_2, ..., v_m be in H. Define H_m to be the span of these vectors (i.e., all possible linear combination of v_i's). For a vector $x \in H$, its orthogonal projection onto the span of v_i's is defined as the vector $\hat{x} \in H_m$ such that one of the following (equivalent) properties holds: (1) the error (i.e., $x - \hat{x}$) is orthogonal to H_m, and (2) among all possible vectors in H_m, $\hat{x}$ is the one that minimizes the norm of the error (i.e., $\|x - \hat{x}\|$ is minimized).

In the discussion above, orthogonality is defined by having the inner product to be zero and the definition of the norm used here is the square root of the inner product of a vector by itself (i.e., (2.1)). As a result, everything here depends on the specific definition of the inner product used.

Solution: Since $\hat{x} \in H_m$, it must be a combination of v_i's. Therefore

$$\hat{x} = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m \tag{2.2}$$

for some set of a_i's. We are looking for the set of a_i's that result in $\hat{x}$ becoming the orthogonal projection. We will use the first property: For each v_i we must have

$$\langle v_i, x - \hat{x} \rangle = 0 \;\Rightarrow \langle v_i, \hat{x} \rangle = \langle v_i, x \rangle.$$

Using (2.2) for $\hat{x}$ on the left hand side, we get

$$\langle v_i, a_1 v_1 \rangle + \langle v_i, a_2 v_2 \rangle + \cdots \langle v_i, a_m v_m \rangle = \langle v_i, x \rangle. \tag{2.3}$$

As a result, we have m equations of the form in (2.3), for v_1, v_2, $\cdots v_m$. These m-linear equations can be solved for the unknowns, a_i's. Stacking these m equations on top of one another, we can form a matrix equation of the form

$$Aa = b$$

where A is a matrix whose (i,j) entry is $\langle v_i, v_j \rangle$, a is the vector of a_i's, and b is a m-vector whose j^{th} entry is $\langle v_j, x \rangle$.

If the vector x and basis v_i's are known, then the development above can be used to solve for a_i's and form the projection $\hat{x} = \sum a_i v_i$.

Remark 2.18. *The matrix A is often called the gramian matrix. It can be shown that it is nonsingular if and only if vectors v_i's are linearly independent. Also, note that we have not assumed that v_i's belong to $\mathcal{R}^n$ or similar spaces. Indeed, the set up here applies to a great many problems. (Think about Fourier series! The basis are orthogonal to one another - in L_2 inner product- so A will be diagonal! Can you finish this ?)*

Definition 2.19. *The relationship $\hat{x} = Px$ defines the projection operator P.*

Remark 2.20. *It can be proven that $P^2 = P$ for all projections (the so called idempotency property) and $P = P^*$ for all orthogonal projections (self adjoint property).*

When applicable, we can find the matrix representation of this operator. Consider the following example where x and all of the v_i's are in $\mathcal{R}^n$

Example 2.21. *For $x, v_i \in \mathcal{R}^n$, define $V = [v_1 \ v_2 \dots v_m]$, which is a $n \times m$ matrix. Then, $A = V^T V$ and $b = V^T x$, if we use $\langle x, y \rangle = x^T y$ for inner product. Then we have,*

$$\hat{x} = Va = V(V^T V)^{-1} V^T x = Px \quad (i.e., \ P = V(V^T V)^{-1} V^T \ )$$

where P is the matrix representation of the projection operator (onto the span of v_i's).

Remark 2.22. *The previous exercise is the famous least squares solution. The next level up would be weighted least squares, which is done by modifying the inner product. For example, by using $\langle x_1, x_2 \rangle = x_1^T W x_2$.*

2.4 Rank, Range, Null Space, etc

Let us start with linear operator A (from $\mathcal{X}$ to $\mathcal{Y}$)

Definition 2.23. *Range of A is the linear space defined by*

$$R(A) = \{y : \exists\ x\ s.t.\ y = Ax\} = A(\mathcal{X}).$$

Definition 2.24. *The null space of A, or kernel of A, is the space defined by*

$$N(A) = \{x \in \mathcal{X}\ s.t.\ Ax = 0\}.$$

A very important property of these spaces is the following

$$R(A) \perp N(A^*) \quad , \quad R(A^*) \perp N(A) \tag{2.4}$$

where A^* is the adjoint consistent with the inner products used. So far, everything we have talked about applies to all linear operators. We can do more for matrices:

Definition 2.25. *For a given $m \times n$ matrix, we have*

$$R(A) = span\ of\ columns\ of\ A$$

$$\rho(A) = rank\ of\ A = dimension\ of\ R(A)$$

$$\rho(A) = min\{\#of\ indep\ rows\ of\ A\ , \#of\ indep\ columns\ of\ A\ \}$$

$$\nu(A) = nulity\ of\ A = dimension\ of\ N(A)$$

Remark 2.26. *An important property of matrices are that for any $A \in \mathcal{C}^{m\times n}$*

$$\rho(A) + \nu(A) = n$$

consequently,

$$\mathcal{R}^n = R(A^*) \overset{\perp}{\oplus} N(A)$$

$$\mathcal{R}^m = R(A) \overset{\perp}{\oplus} N(A^*)$$

2.5 Stability and Related Topics

Consider the dynamical system described by

$$\dot{x} = Ax \tag{2.5}$$

if A is time varying, eigenvalues tell you *next to nothing* about the stability of this system. Recall that the study of stability is, typically, the study of equilibrium points, which for linear systems boil down to study of the point $x = 0$. The most common way to study the stability of such systems is through the 'Lyapunov second - or direct- method'. Its most important results can be summarized by the following:

- If $\exists$ a continuously differentiable function $V(x)$ that has the following properties , (1) $V(x) > 0$ for all nonzero x, $V(x)$ is radially unbounded (i.e, as x gets large, so does V), and (3) $\dot{V} < 0$ for all nonzero x, then the system in (2.5) is globally asymptotically stable.
- If we could only guarantee $\dot{V} \leq 0$, then it is marginally stable, or stable in the sense of Lyapunov. If $\dot{V}$ can change sign, then the system is not stable.

Remark 2.27. *This is a sufficient condition only. Also, a great deal more can be said about this method. We leave the details and embellishments to the study of nonlinear systems.*

Remark 2.28. *In most applications, the Lyapunov function is chosen to be $V(x) = x^T P x$ for some $P > 0$. While we discuss the notion of positive definiteness later in some details, for now it suffices to know that $P > 0$ means that P is positive definite, which means that for* any *non-zero x, $x^T P x > 0$.*

If the matrix A in (2.5) is constant, a great deal of simplification can be made:

- If the real parts of every eigenvalue of A is strictly negative (i.e., it is on the open left half plane), then the system is globally asymptotically (and exponentially) stable. (i.e, starting with any x_o, $x(t)$ will go to zero as time goes by).
- If the real part of one or more eigenvalues is zero then the system is at best marginally stable (stability in the sense of Lyapunov). In particular, if zero real parts correspond to Jordan blocks of dimension 2 or more, then the system is *unstable*.

A few odds and ends should be reviewed. Recall that for constant A, the solution of (2.5) is $x(t) = e^{A(t-t_o)}x_o$ where x_o is the state at time t_o and $e^{A(t-t_o)}$ is the state transition matrix. Also, recall that if all eigenvalues of A have

negative real parts, then as $t \to \infty$, $e^{A(t-t_o)} \to 0$. Indeed, its norm is bounded by $Me^{\alpha(t-t_o)}$, where M is a fixed constant and α is the real part of the least stable eigenvalue. As a result, integrals of the form

$$\lim_{t\to\infty} \int_0^t e^{At}\, dt$$

remain bounded, and converge, if and only if A is stable (i.e., real part λ is strictly negative).

2.5.1 Lyapunov Equation

Consider the following linear equation

$$PA + A^T P = -Q \tag{2.6}$$

where both P and Q are symmetric. From now on, by stable we mean all eigenvalues in the open left half plane. The basic results can be summarized

- A is stable iff for every $Q > 0$, (2.6) has a unique positive definite solution
- A is stable iff for every $Q \geq 0$ and (A, Q) observable, (2.6) has a unique positive definite solution

Proof: (Sketch)
$\Rightarrow P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$ is the solution to the Lyapunov equation. Check with Leibniz rule, show uniqueness of the solution, prove positive definiteness.
$\Leftarrow$ Use $V(x) = x^T P x$ and show $\dot{V} < 0$, where you may end up using LaSalle's lemma!

2.6 Controllability and Observability

In this subsection, we will focus on different definitions of observability for linear time invariant systems. Recall that controllability is dual of observability and all relevant results can be obtained by replacing A with A^T and C with B^T. We start with

$$\dot{x} = Ax + Bu \quad , \quad y = Cx \tag{2.7}$$

where A is a $n \times n$ matrix and B and C are matrices of appropriate dimension. The system in (2.7) is observable if any of the following equivalent conditions hold

1. Given $u(t)$ and $y(t)$ for $t \in [0, T]$, $x(0)$ can be determined.
2. For $u(t) \equiv 0$, $Ce^{At}x_o = 0$ for any interval implies the initial condition was zero ($x_o = 0$).
3. For $u(t) \equiv 0$, $W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau}\, d\tau \ > 0$, for any $t > 0$
4. rank $\begin{pmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{pmatrix} = n$
5. rank $\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} = n$ for all complex λ
6. $Ax = \lambda x$ and $Cx = 0$, (together) imply $x = 0$

Definition 2.29. *The system in* (2.7) *is detectable if all of the unstable modes are observable; i.e., $Ax = \lambda x$ and $Cx = 0$ imply either $x = 0$ or $Real(\lambda) < 0$. Similarly, the system in* (2.7) *is stabilizable if all of the unstable modes are controllable; i.e., $A^T x = \lambda x$ and $B^T x = 0$ imply either $x = 0$ or $Real(\lambda) < 0$. Some books define stabilizability as: The system is stabilizable if and only if there exists a matrix K, of appropriate dimension, such that $A - BK$ is stable (strictly). Similar definitions also hold for detectability.*

2.7 PROBLEM SET

Exercise 2.30. *Prove the famous Cauchy-Schwarz inequality:*

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}}$$

where equality holds only when one of the vectors is zero or the vectors are linearly dependent.

Exercise 2.31. *Confirm* (2.1) *is an appropriate norm.*

Exercise 2.32. *Confirm that the inner products in* example 2.7 *and* example 2.8 *are proper inner products. What is the corresponding norm in each case?*

Exercise 2.33. *Show that for any $x \in \mathcal{R}^n$, we have*

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$$

Exercise 2.34. *Show that the operator norm satisfies the norm properties. Furthermore, show that $\|AB\|_i \leq \|A\|_i\|B\|_i$. This property is not true for non-induced norms of linear operators!*

Exercise 2.35. *In* example 2.15*, prove the expression given for induced 1-norm and ∞-norm are correct.*

Exercise 2.36. *Show that the gramian matrix encountered in orthogonal projection is nonsingular iff the basis vectors are linearly independent.*

Exercise 2.37. *In orthogonal projection of a vector in $\mathcal{R}^n$ onto the span of other vectors (also in $\mathcal{R}^n$), show that: (a) If $x \in H_n$ then $x = \hat{x}$, (b) $P = P^2$ and $P = P^*$ (i.e., is self-adjoint), and (c) $(I - P)$ is also an orthogonal projection.*

Exercise 2.38. *Show that the property* (2.4) *is true.*

Exercise 2.39. *Provide a compete proof of the Lyapunov equation, for the case of $Q > 0$. First, show that if the equation holds for $P > 0$ and $Q > 0$, then A must be stable. Next, show that if the equation holds, A is stable and $Q > 0$, then $P > 0$. (NOTE: it is not true that if A is stable and $P > 0$, the resulting Q is necessarily positive definite!).*

Exercise 2.40. *Write the controllability equivalence of properties 1-6 of the observability.*

Exercise 2.41. *In the observability properties, show 5 $\Leftrightarrow$ 6 and 2 $\Leftrightarrow$ 3.*

Exercise 2.42. *Given that the system in* (2.7) *is observable and given $y(t)$ and $u(t)$, how would you calculate $x(0)$? What is the dual problem in controllability?*

Exercise 2.43. *For a stable A, prove that the observability matrix $W_o(\infty)$ in property 3 satisfies $PA + A^T P = -C^T C$*

3 Eigenvalues, Singular Values and Pseudo inverse.

3.1 Eigenvalues and Eigenvectors

For a square $n \times n$ matrix A, we have the following definition:

Definition 3.1. *If there exist (possibly complex) scalar λ and vector x such that*

$$Ax = \lambda x, \quad \textit{or equivalently}, \quad (A - \lambda I)x = 0, \quad x \neq 0$$

then x is the eigenvector corresponding to the eigenvalue λ. Recall that any $n \times n$ matrix has n eigenvalues (the roots of the polynomial $det(A - \lambda I)$).

Definition 3.2. *Matrix A is called* simple *if it has n linearly independent eigenvectors.*

Definition 3.3. *Let $A^H \triangleq \bar{A}^T$, $x^H \triangleq \bar{x}^T$ (i.e., complex conjugate transpose). Matrix A is:*
Hermitian if $A = A^H \Leftrightarrow x^H Ax = real$, for all $x \in C^n$
Normal if $AA^H = A^H A$
Unitary if $AA^H = A^H A = I$
Orthogonal if $AA^T = A^T A = I$, (for A real)

Definition 3.4. *Hermitian matrix D (i.e., $D = D^H$) is*
positive definite if $x^H Dx > 0$ for all $x \neq 0$
positive semi definite if $x^H Dx \geq 0$ for all $x \neq 0$
negative definite if $x^H Dx < 0$ for all $x \neq 0$
negative semi definite if $x^H Dx \leq 0$ for all $x \neq 0$
indefinite if $x^H Dx < 0$ for some nonzero x and $x^H Dx > 0$ for some other nonzero x

Definition 3.5. *If $A = QBQ^{-1}$, for some nonsingular Q, then 'A is* similar *to B' or B is obtained via a similarity transformation (Q) of A. If we had $A = QBQ^T$, then A is obtained through a 'congruent' transformation on B.*

P1. For *general* matrix A: If all e-values are distinct; i.e., $\lambda_i \neq \lambda_j$, $(i \neq j)$, then A has n linearly independent eigenvectors; i.e., it is *simple*. Furthermore, we have

$$A = Q\Lambda Q^{-1}, \quad \Lambda = Q^{-1}AQ$$

where $Q = [x_1 \dots x_n]$ (the e-vectors) and Λ is a diagonal matrix with λ_i on the (i,i) element. (Such a matrix is sometimes called *Diagonalizable*) .

P2. For Hermitian D, its eigenvalues are real; i.e, $Imag(\lambda_i) = 0 \ \ \forall i$. Furthermore, if D is real (i.e., real symmetric) the eigenvectors are real as well.

P3. If D is Hermitian, it is also *simple* .

P4. For $D = D^H$ (i.e, Hermitian D) eigenvectors corresponding to distinct eigenvalues are orthogonal in the sense that $x_j^H x_i = 0,\ if\ \lambda_i \neq \lambda_j$.

P5. For $D = D^H$, let $x_1 \cdots x_m$ be the eigenvector corresponding to the repeated eigenvalue $\hat{\lambda}$. Show that if we replace the $x_i's$ with their Gramm-Schmidt vectors, we still have m eigenvectors for $\hat{\lambda}$.

P6. For Hermitian D, the eigenvector matrix can be written as a unitary matrix; that is

$$D = Q\Lambda Q^H, \quad QQ^H = Q^H Q = I,\ ,\Lambda\ real, \quad Q\ real\ if\ D\ real\ symmetric$$

P7. If $D = D^H$ is positive (semi) definite, then $D_{ii} > (\geq)0$, with similar result for negative (semi) definite.

P8. For a Hermitian matrix D, we have
D positive semi definite if and only if (iff or $\Longleftrightarrow$) $\lambda_i \geq 0, \quad \forall i$
D is positive definite iff $\lambda_i > 0, \quad \forall i$
D is negative semi definite iff $\lambda_i \leq 0, \quad \forall i$
D is negative definite iff $\lambda_i < 0, \quad \forall i$
D is indefinite iff $\lambda_i > 0$ for some i and $\lambda_i < 0$ for some other i

P9. For any matrix A, $x^H A^H A x \geq 0, \quad \forall x$. Sometimes we write $A^H A \geq 0$ for short.

P10. If Hermitian D is positive semi definite ($D \geq 0$), then there exist Hermitian matrices V such that

$$D = VV,\ ;\ \ e.g.,\ \ V = Q(\Lambda)^{0.5}Q^H$$

and furthermore there exist matrices C such that

$$D = C^H C\ ;\ \ e.g.,\ \ C = (\Lambda)^{0.5}Q^H$$

P11. If Q is unitary, all of its eigenvalues have magnitude one; i.e, $|\lambda_i(Q)| = 1$.

P12. If λ is an eigenvalue of A, it is also an eigenvalue of A^T. Also, $\bar{\lambda}$ is an eigenvalue of A^H. Therefore if A is real, eigenvalues appear in complex conjugate pairs.

P13. If A is normal, then

$$Ax = \lambda x \Longleftrightarrow A^H x = \bar{\lambda} x$$

P14. If A is normal, its eigenvectors are orthogonal, in the sense that $x_i^H x_j = 0$

P15. If $A^2 = A$ then all eigenvalues of A are either zero or one (idempotent matrix)

P16. If $A^k = 0$ for any integer k, then all eigenvalues of A are zero (nilpotent matrix)

P17. For any Hermitian matrix D

$$\lambda_{min}(D)x^H x \leq x^H D x \leq \lambda_{max}(D)x^H x \quad \forall x \in C^n$$

where λ_{min} is the smallest eigenvalue (algebraically). This inequality is often called Raleigh's inequality.

P18. For any two Hermitian matrices M and N,

$$\lambda_{min}(M+N) \geq \lambda_{min}(N)+\lambda_{min}(M) \text{ , } and \text{ } \lambda_{max}(M+N) \leq \lambda_{max}(N)+\lambda_{max}(M)$$

P19. If (λ, x) are an eigenvalue/eigenvector pair of the matrix AB, with $\lambda \neq 0$, then (λ, Bx) is an eigenvalue/eigenvector pair for BA.

P20. If A and B are similar (via transformation Q), they have the same eigenvalues and their eigenvectors differ by a Q term.

3.2 Singular Value Decomposition (SVD)

For the development below, assume $A \in C^{m\times n}$, $m \geq n$, with rank r (i.e., $\rho(A) = r$). Note that $A^H A \in C^{n\times n}$ and $AA^H \in C^{m\times m}$. Also, for inner product and norm, we use $\|x\|^2 =< x, x >$, with $< x, y >= x^H y$.

We need to review the following properties

$$Range(A) = Range(AA^H), \quad and \quad Range(A^H) = Range(A^H A)$$

which implies $\rho(A) = \rho(A^H) = \rho(AA^H) = \rho(A^H A) = r$. The basic SVD can be obtained through the following

SVD1. Let $AA^H u_i = \sigma_i^2 u_i$, for $i = 1, 2, \cdots m$.

$$U \stackrel{\triangle}{=} [u_1 \;\; u_2 \;\; \cdots u_m], \;\; U \in C^{m\times m}, \quad UU^H = U^H U = I_m.$$

We then have $\|A^H u_i\| = \sigma_i$ for $i = 1, 2, \cdots m$.

SVD2. Let $A^H A v_i = \hat{\sigma}_i^2 v_i$, for $i = 1, 2, \cdots n$, such that

$$V \stackrel{\triangle}{=} [v_1 \;\; v_2 \;\; \cdots v_n], \quad V \in C^{n\times n}, \quad VV^H = V^H V = I_n.$$

Then nonzero $\hat{\sigma}_i$'s are equal to nonzero σ_i's of SVD1, with $v_i = \frac{A^H u_i}{\sigma_i}$. For zero $\hat{\sigma}_i$, we have $Av_i = 0$. (To show this, use P19 of the eigenvalue handout. Show that $A^H A$ and AA^H have the same nonzero eigenvalues, with $v's$ as defined above). These v_i's are linearly independent and form a set of orthonormal vectors.

SVD3. Consider the following n equations for $i = 1, 2, \cdots n$:

$$Av_i = AA^H \frac{u_i}{\sigma_i} \;\; (or \; zero) = \sigma_i u_i \; (or \; zero).$$

These equations can be written as

$$AV = U\Sigma, \;\; \Longleftrightarrow \;\; A = U\Sigma V^H \tag{3.1}$$

where U and V are the same as SVD1 and SVD2, respectively. Σ is a $m \times n$ matrix, with the top left $n \times n$ block in diagonal form with σ_i's on the diagonal and the bottom $(m - n) \times n$ rows zero. Without loss of any generality, we let $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n \geq 0$. These σ_i's are called the *singular values* of A (or A^H). Since rank of A is assumed to be $r \leq min\{m, n\}$, there are exactly r nonzero singular values (Why? recall SVD1 and SVD2). Therefore, we can write

$$U = [U_r \;\; \bar{U}_r], \quad U_r \in C^{m\times r}, \qquad V = [V_r \;\; \bar{V}_r], \quad V_r \in C^{n\times r}, \tag{3.2}$$

and

$$\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_r = diag\{\sigma_1, \sigma_2, \ldots, \sigma_r\} \tag{3.3}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Or condensing (3.1),

$$A = U_r \Sigma_r V_r^H. \tag{3.4}$$

Equations (3.1) or (3.4) are often called the 'singular value decomposition of A'. If A is a real matrix, all vectors (i.e, u_i's, v_i's) will be real and the superscript 'H' is replaced by 'T' - transpose. We can now discuss some of the main properties of singular values. First we introduce the following notation

$$\overline{\sigma}(A) \stackrel{\triangle}{=} \sigma_{max}(A), \quad \underline{\sigma}(A) \stackrel{\triangle}{=} \sigma_{min}(A), \tag{3.5}$$

where σ_i i the i^{th} singular value. Recall that an $m \times n$ matrix has n singular values, of which the last $n - r$ are zero ($r = \rho(A)$).

P1-SVD. The 'principal gains' interpretation:

$$\overline{\sigma}(A)\|x\|_2 \geq \|Ax\|_2 \geq \underline{\sigma}(A)\|x\|_2, \ \forall x \tag{3.6}$$

P2-SVD. The induced 2-norm:

$$\overline{\sigma}(A) = \|A\|_2 = sup \frac{\|Ax\|_2}{\|x\|_2}, \quad x \neq 0. \tag{3.7}$$

P3-SVD. If A^{-1} exists,

$$\overline{\sigma}(A) = \frac{1}{\underline{\sigma}(A^{-1})}. \tag{3.8}$$

Extra1. Null space of $A = \text{span}\{v_{r+1} \cdots v_n\}$ and range space of $A = \text{span}\{u_1 \cdots u_r\}$.

Extra2. $U_r^H U_r = I_r$ and $U_r U_r^H$ is the orthogonal projection operator onto the Range of A. (recall $R(A) = R(AA^H)$, but $R(AA^H) = span(u_1, \cdots, u_r)$, since u_i's are orthonormal, direct calculation of the projection operator gives the result).

Extra3. $V_r^H V_r = I_r$ and $V_r V_r^H$ is the orthogonal projection operator onto the Range of A^H.

3.3 A Famous Application of SVD

Let us consider the equation

$$Ax_o = b_o \quad \Rightarrow x_o = A^{-1}b_o$$

assuming that the inverse exists and A is known accurately. Now let there be some error in our data; i.e., let $b = b_o + \delta b$, where δb is the error or noise, etc. Therefore, we are now solving

$$Ax = b_o + \delta b \;\Rightarrow x = A^{-1}b_o + A^{-1}\delta b = x_o + \delta x.$$

We are interested in investigating how small or large is this error in the answer (i.e., δx) for a given amount of error. Note that

$$\delta x = A^{-1}\delta b \;\Rightarrow \|\delta x\| \leq \|A^{-1}\|\;\|\delta b\|$$

or since $\|A^{-1}\| = \sigma_{max}A^{-1} = \frac{1}{\sigma_{min}A}$, we can write

$$\|\delta x\| \leq \frac{\|\delta b\|}{\sigma_{min}A}. \tag{3.9}$$

However, recall that $x_o = A^{-1}b_o$ and therefore

$$\|x_o\| \geq \sigma_{min}(A^{-1})\|b_o\| = \frac{\|b_o\|}{\sigma_{max}A}. \tag{3.10}$$

Combining (3.9) and (3.10)

$$\frac{\|\delta x\|}{\|x_o\|} \leq \frac{\|\delta b\|}{\sigma_{min}A}\;\frac{1}{\|x_o\|}$$

or

$$\frac{\|\delta x\|}{\|x_o\|} \leq \frac{\|\delta b\|}{\|b_o\|}\;\frac{\sigma_{max}A}{\sigma_{min}A}$$

where the last fraction is called 'the condition number of A'. This number is indicative of the magnification of error in the linear equation of interest. Similar analysis can be done regarding a great many numerical and computational issues. In most problems, a matrix with very large condition number is called ill conditioned and will result in severe numerical difficulties.

Note that by definition, the condition number is equal or larger than one. Also, note that for unitary matrices, the condition number is one (one of the main reasons these matrices are used heavily in computational linear algebra).

3.4 Important Properties of Singular Values

In the following, use $\overline{\sigma}(A)$ as the maximum singular value of A, $\underline{\sigma}(A)$ as the minimum singular value and $\sigma_i(A)$ as the generic i^{th} singular value.

In all cases, $A \in \mathcal{C}^{m\times n}$. Recall that $\sigma_i^2 = \lambda_i(A^HA) = \lambda_I(AA^H)$, and that $\sigma_i(A) \geq 0$.

P4-SVD. $\sigma_i(\alpha A) = |\alpha|\sigma_i(A),\ \forall\alpha \in \mathcal{C}$

P5-SVD. $\overline{\sigma}(AB) \leq \overline{\sigma}(A)\,.\,\overline{\sigma}(B)$

P6-SVD. $\overline{\sigma}(A+B) \leq \overline{\sigma}(A) + \overline{\sigma}(B)$

P7-SVD. $\underline{\sigma}(AB) \geq \underline{\sigma}(A)\,.\,\underline{\sigma}(B)$

P8-SVD. $\underline{\sigma}(A) \leq |\lambda_i(A)| \leq \overline{\sigma}(A)\ \ \forall\ i$

P9-SVD. $\underline{\sigma}(A) - 1 \leq \underline{\sigma}(I+A) \leq \underline{\sigma}(A) + 1$

P10-SVD. $\underline{\sigma}(A) - \overline{\sigma}(B) \leq \underline{\sigma}(A \underline{+} B) \leq \underline{\sigma}(A) + \overline{\sigma}(B)$

P11-SVD. $\overline{\sigma}(A) \leq \sqrt{trace(A^HA)} \leq \sqrt{n}\ \overline{\sigma}(A)$

P12-SVD. $TraceA^HA = \sum_1^k \sigma_i^2(A), \quad k = min(n,m)$

P13-SVD. $detA^HA = \prod_1^k \sigma_i^2(A)$

P14.-SVD In general, $\sigma_i(AB) \neq \sigma_i(BA)$

P15-SVD. $\overline{\sigma}(A)\underline{\sigma}(B) \leq \overline{\sigma}(AB) \quad A \in \mathcal{C}^{m\times n},\ \ B \in \mathcal{C}^{n\times l}\ \ n \leq l\ only$
– $\overline{\sigma}(B)\underline{\sigma}(A) \leq \overline{\sigma}(AB) \quad A \in \mathcal{C}^{m\times n},\ \ B \in \mathcal{C}^{n\times l}\ \ n \leq m\ only$

P16-SVD. $\underline{\sigma}(AB) \leq \overline{\sigma}(A)\underline{\sigma}(B) \quad no\ restrictions$
– $\underline{\sigma}(AB) \leq \overline{\sigma}(B)\underline{\sigma}(A) \quad no\ restrictions$

P17-SVD. $\underline{\sigma}(A)\underline{\sigma}(B) \leq \underline{\sigma}(AB) \leq \overline{\sigma}(A)\underline{\sigma}(B) \leq \overline{\sigma}(AB) \leq \overline{\sigma}(B)\overline{\sigma}(A),\ \ n \leq l$
– $\underline{\sigma}(A)\underline{\sigma}(B) \leq \underline{\sigma}(AB) \leq \overline{\sigma}(B)\underline{\sigma}(A) \leq \overline{\sigma}(AB) \leq \overline{\sigma}(B)\overline{\sigma}(A),\ \ n \leq m$

3.5 Pseudo Inverse

The basic definition of inverse of a matrix A is well known, when it is square and full rank. For non-square, but full rank, matrix $A \in R^{m\times n}$, we have the following: When $m > n$ ($n > m$) left (right) inverse of A is the matrix B in $R^{n\times m}$ (in $R^{m\times n}$) such that BA (AB) is I_n (I_m).

When the matrix is not full rank, the so called 'pseudo' inverses are used. The famous definition of Penrose is the following. The pseudo inverse of A is the unique matrix (linear operator) $A^\dagger$ that satisfies the following

1. $(A^\dagger A)^H = A^\dagger A$
2. $(AA^\dagger)^H = AA^\dagger$
3. $A^\dagger AA^\dagger = A^\dagger$
4. $AA^\dagger A$=A

Recalling that matrix P is a projection if $P^2 = P$ and is orthogonal projection if $P = P^2$ *and* $P = P^H$, we can see that the pseudo inverse has the following properties

- $A^\dagger A$ is the orthogonal projection onto Range of A^H
- $AA^\dagger$ is the orthogonal projection onto Range of A
- $(A^\dagger)^\dagger = A$

Now we will suggest the following candidate:

$$A = U_r\Sigma_r V_r^H \Longrightarrow A^\dagger = V_r\Sigma_r^{-1}U_r^H \tag{3.11}$$

PINV1. Show that for full rank matrices, the definition in (3.11) reduces to standard inverse (square matrices) or left or right inverse.

PINV2. Verify that $A^\dagger$ defined in (3.11) satisfies the basic properties of pseudo inverse.

To gain a better understanding of the pseudo inverse, consider the linear equation $Ax = y$. When A is square and full rank, the solution is $A^{-1}y$. In general, we say that the least squares solution of this problem is $A^\dagger y$! Let us investigate some more.

PINV3. Show that when A is a wide (or long) matrix with full row rank, the problem has infinitely many solutions, among which only one is in the range of A^H. Further, this solution has the smallest norms among all possible solutions. The solution is $x = (right\ inverse\ of\ A)y$.

PINV4. When A is a tall matrix with full column rank, then $x = (\text{ left inverse of } A)\ y$ gives the unique solution or (if no solution exists) the solution that minimizes the 2-norm of the error $(y - Ax)$.

We can generalize this by letting A be rank deficient. Starting with y, we find y_p its projection onto range of A to minimize the norm of the error ($y_p = y$ if at least one solution exists). Now $Ax = y_p$ has one or many solutions, among which the one with minimum norm is the *unique* vector x_o such that it is in the range space of A^H. The relationship between x_o and y is $x_o = A^\dagger y$. In short, the pseudo inverse simultaneously minimizes the norm of the error as well as the norm of the solution itself.

PINV5. Show that the definition of $A^\dagger$ in (3.11) is the same as the development discussed above (i.e., show that Ax_o is equal to y_p and x_o is in the range of A^H. For this last part recall that the range of A^H is the same as range of $A^H A$ which is the same as span of the v_1 to v_r).

Another common, and equivalent, definition (see Zadeh and Desoer) for the pseudo inverse is the matrix satisfying

1. $A^\dagger A x = x \quad \forall x \in range\ of A^H$
2. $A^\dagger z = 0 \quad \forall z \in null\ space\ of A^H$
3. $A^\dagger(y + z) = A^\dagger y + A^\dagger z \quad \forall y \in R(A) \quad , \forall z \in R(A)^\perp$

Finally, they suggest the following calculation for the inverse

$$A^\dagger = (A^H A)^\dagger A^H \tag{3.12}$$

PINV6. Show that (3.12) results in the same matrix as (3.11).

4 THE LINEAR QUADRATIC REGULATOR

In this Section, we will deal with the 'Linear Quadratic Regulator' problem (or LQR for short). We start with the most general from; that of time varying system matrices and finite horizon.

4.1 Time varying and finite horizon case

Consider the dynamical system

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ x(t_o) = x_o \end{cases} \tag{4.1}$$

We are interested in finding a control $u(t)$ such that the following cost functional is minimized

$$J(t_o, t_f, x_o, u(.)\,) = \int_{t_o}^{t_f} \{\, x^T(t)Q(t)x(t) + u^T(t)R(t)u(t),\ \}\, dt \ +\ x^T(t_f)\, P_1\, x(t_f) \tag{4.2}$$

where

$$\begin{cases} t_f \textit{ is the fixed final time} \\ P_1 \geq 0 \textit{ is the terminal penalty term} \\ Q(t) \geq 0\ \forall t \in [0, t_f] \\ R(t) > 0\ \forall t \in [0, t_f] \end{cases} \tag{4.3}$$

The desired solution would give us a control law for $u(.)$. This may or may not be feedback (or even linear). While there are several ways to approach this problem, we will use the perturbation or variation approach. For this, we make the following *assumption*:

Assumption 4.1. *Suppose there exists an optimal control law $u^*(.)$ that minimizes* (4.2)*, subject to* (4.1)*.*

Therefore, any other control law cannot do better! Now let us implement $u^*(t)$, and label the resulting trajectory, which minimizes (4.2), $x^*(t)$; i.e.,

$$\begin{cases} \dot{x}^*(t) = A(t)\,x^*(t) + B(t)\,u^*(t) \\ x^*(t_o) = x_o \end{cases} \tag{4.4}$$

Now *all other* control laws can be represented by

$$u(t) = u^*(t) + \epsilon\,\tilde{u}(t), \quad t \in [0, t_f] \tag{4.5}$$

where ϵ is a (possibly negative) scalar and $\tilde{u}(t)$ is the control perturbation (function of time). Note that if $u(t)$ is implemented, the resulting trajectory will be the $x(t)$ of (4.1).

At this point we can introduce the state perturbation $\tilde{x}(t)$

$$\epsilon\tilde{x}(t) \stackrel{\triangle}{=} x(t) - x^*(t) \Longleftrightarrow x(t) = x^*(t) + \epsilon\tilde{x}(t) \tag{4.6}$$

where ϵ is the same as in (4.5) and $x(t)$ is the response to the control $u(t)$ - as in (4.5). Combining (4.5) and (4.6), we get

$$\begin{cases} \dot{\tilde{x}}(t) = A(t)\,\tilde{x}(t) + B(t)\,\tilde{u}(t) \\ \tilde{x}(t_o) = 0. \end{cases} \tag{4.7}$$

From the first class in linear systems, the solution (or response) to (4.7) is

$$\tilde{x}(t) = \int_{t_o}^{t} \Phi(t,\tau)\,B(\tau)\tilde{u}(\tau)\,d\tau \tag{4.8}$$

where $\Phi(t,\tau)$ is the state transition matrix associated with $A(t)$. Recall that state transition matrix satisfies the following

$$\begin{cases} \Phi^{-1}(t,\tau) = \Phi(\tau,t) \\ \Phi(t,t) = I \\ \Phi(t,t_1)\,\Phi(t_1,t_2) = \Phi(t,t_2) \quad \forall t_1 \in [t,t_2] \\ \frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau) \end{cases} \tag{4.9}$$

Let us go back to our problem. Since $u^*(.)$ is the optimal control, no ϵ or $\tilde{u}(.)$ can result in a smaller cost function than the following optimal one

$$\begin{aligned} J_{min} &= J(t_o,t_f,x_o,u^*(.)\,) \qquad (4.10)\\ &= \int_{t_o}^{t_f} \{\, x^{*T}(t)Q(t)x^*(t) + u^{*T}(t)R(t)u^*(t)\,\}\,dt + x^{*T}(t_f)\,P_1\,x^*(t_f). \end{aligned}$$

Therefore, if we calculate the cost due to a non-optimal control law, we expect to have a larger than (or at best equal to) J_{min}. That is, if we implement some $u(t) = u^*(t) + \epsilon\,\tilde{u}(t)$, the cost will be

$$\begin{aligned} J(t_o,t_f,x_o,u(.)\,) &= \int_{t_0}^{t_f} \{\,[x^*(t) + \epsilon\tilde{x}(t)]^T Q(t)[x^*(t) + \epsilon\tilde{x}(t)]\,\}\,dt \\ &+ \int_{t_0}^{t_f} \{\,[u^*(t) + \epsilon\tilde{u}(t)]^T R(t)[u^*(t) + \epsilon\tilde{u}(t)]\,\}\,dt \\ &+ [x^*(t_f) + \epsilon\tilde{x}(t_f)]^T P_1[x^*(t_f) + \epsilon\tilde{x}(t_f)] \qquad (4.11) \end{aligned}$$

which cannot be any less that J_{min}. Now let us rearrange (4.11) and group in terms of powers of ϵ.

$$
\begin{aligned}
& J(t_o, t_f, x_o, u(.)\,) \\
= & \int_{t_0}^{t_f} \{\, x^{*T}(t)Q(t)x^*(t) + u^{*T}(t)R(t)u^*(t)\,\}\, dt \;+\; x^{*T}(t_f)\, P_1\, x^*(t_f) \\
+ & \; 2\epsilon \left[\int_{t_0}^{t_f} \{\, x^{*T}(t)Q(t)\tilde{x}(t) + u^{*T}(t)R(t)\tilde{u}(t)\,\}\, dt \;+\; x^{*T}(t_f)P_1\tilde{x}(t_f) \right] \\
+ & \; \epsilon^2 \left[\int_{t_0}^{t_f} \{\, \tilde{x}^T(t)Q(t)\tilde{x}(t) + \tilde{u}^T(t)R(t)\tilde{u}(t)\,\}\, dt \;+\; \tilde{x}^T(t_f)P_1\tilde{x}(t_f) \right] \quad (4.12)
\end{aligned}
$$

This expression holds for all possible $\tilde{u}(.)$ and all possible ϵ. Equation (4.12) has the form of

$$J = A + \epsilon B + \epsilon^2\, C$$

where A is the minimum of J and B and C are independent of ϵ and $C \geq 0$ (all scalars). As discussed in the homework problems, it follows that B, the coefficient of the ϵ term, must be zero (it is a necessary condition). That is

$$\int_{t_0}^{t_f} \{\, x^{*T}(t)Q(t)\tilde{x}(t) + u^{*T}(t)R(t)\tilde{u}(t)\,\}\, dt \;+\; x^{*T}(t_f)P_1\tilde{x}(t_f) = 0, \quad \forall\ \tilde{u}. \quad (4.13)$$

Now use (4.8) for $\tilde{x}$ in the above equation to obtain

$$
\begin{aligned}
& \int_{t_0}^{t_f} \left[x^{*T}(t)Q(t) \int_{t_o}^{t} \Phi(t,\tau)B(\tau)\tilde{u}(\tau)d\tau \;\right] dt + \int_{t_o}^{t_f} u^{*T}(t)R(t)\tilde{u}(t)\; dt \\
+ & \quad x^{*T}(t_f)P_1 \int_{t_o}^{t_f} \Phi(t_f,t)B(t)\tilde{u}(t)dt = 0, \quad \forall\ \tilde{u}.
\end{aligned}
$$

We will concentrate on the first term. This term can be manipulated in the following form (through basic change of variables)

$$
\begin{aligned}
& \int_{t_0}^{t_f} \int_{t_0}^{t} x^{*T}(t)Q(t)\Phi(t,\tau)B(\tau)\tilde{u}(\tau)d\tau\; dt \overset{\tau \leftrightarrow t}{=} \\
& \int_{t_0}^{t_f} \int_{t_0}^{\tau} x^{*T}(\tau)Q(\tau)\Phi(\tau,t)B(t)\tilde{u}(t)dt\; d\tau
\end{aligned}
$$

which, according to yet another homework problem (!), can be written as

$$\int_{t_0}^{t_f} \int_{t}^{t_f} x^{*T}(\tau)\, Q(\tau)\, \Phi(\tau,t)\, B(t)\, \tilde{u}(t)\;\; d\tau\; dt.$$

Incorporating all in (4.13) we get the following necessary condition for $u^*(.)$ to be optimal

$$\int_{t_0}^{t_f} \left[\int_t^{t_f} x^{*T}(\tau)\, Q(\tau)\, \Phi(\tau,t)\, B(t)\ d\tau + u^{*T}(t)\, R(t) \right] \tilde{u}(t)\ dt$$
$$+ \quad \int_{t_0}^{t_f} \left[x^{*T}(t_f)\, P_1\, \Phi(t_f,t) B(t) \right] \tilde{u}(t)\ dt = 0. \tag{4.14}$$

Recall that equation (4.14) holds for all possible $\tilde{u}(.)$, which implies that the integrand must be zero, identically. (Technically, we should say almost everywhere!). Therefore, we have

$$\int_t^{t_f} x^{*T}(\tau)\, Q(\tau)\, \Phi(\tau,t)\, B(t)\ d\tau + u^{*T}(t)\, R(t) + x^{*T}(t_f) P_1 \Phi(t_f,t) B(t) = 0. \tag{4.15}$$

Next, we will define the (so called co-state) vector $p(t)$ according to

$$p^T(t) \triangleq \int_t^{t_f} x^{*T}(\tau)\, Q(\tau)\, \Phi(\tau,t)\ d\tau + x^{*T}(t_f) P_1 \Phi(t_f,t). \tag{4.16}$$

With this definition, (4.15) gives an expression for the optimal control law, since (4.15) and (4.16) imply

$$u^{*T}(t)\, R(t) + p^T(t) B(t) = 0 \Rightarrow u^{*T}(t) = -p^T(t) B(t) R^{-1}(t)$$

or

$$u^*(t) = -R^{-1}(t)\, B^T(t)\, p(t). \tag{4.17}$$

Note that we are still far from done. We need to find $p(t)$ and even then the control law appears to be open loop!! Now with the help of Leibniz rule (see homework problems!) we can find the derivative of $p(t)$ the so called co-state vector.

$$\begin{cases} \dot{p}(t) = -Q(t)\, x^*(t) - A^T(t)\, p(t) \\ p(t_f) = P_1 x^*(t_f). \end{cases} \tag{4.18}$$

Equation (4.18) is called the adjoint equation. Now using (4.17) in (4.4), we get the following set of differential equations

$$\begin{cases} \dot{x}^*(t) = A(t)\, x^*(t) - B(t)\, R^{-1}(t)\, B^T(t)\, p(t) \\ \dot{p}(t) = -Q(t)\, x^*(t) - A^T(t)\, p(t) \end{cases} \tag{4.19}$$

with the end condition

$$\begin{cases} x^*(t_o) = x_o \\ p(t_f) = P_1 x^*(t_f) \end{cases} \tag{4.20}$$

Equations (4.19) and (4.20) constitute a set of 'two point boundary value' problem. A common way to approach this problem, i.e., the fact that $p(t_o)$ is not known, is to employ trial and error type techniques. In any rate, solving this set of equations can be formidable. Worse yet, it still leaves us with an open loop control!! These two problems force us to dig deeper!

We will go back to (4.19) and try to write the response! From (4.19) we have

$$\begin{pmatrix} x^*(t_f) \\ p(t_f) \end{pmatrix} = \Theta(t_f, t) \begin{pmatrix} x^*(t) \\ p(t) \end{pmatrix}$$

where $\Theta(t_f, t)$ is the state transition matrix corresponding to the $2n$ order system of (4.19). Using the semi-group property of the state transition matrix, we can write

$$\begin{pmatrix} x^*(t) \\ p(t) \end{pmatrix} = \Theta(t, t_f) \begin{pmatrix} x^*(t_f) \\ p(t_f) \end{pmatrix} = \begin{pmatrix} \theta_{11}(t, t_f) & \theta_{12}(t, t_f) \\ \theta_{21}(t, t_f) & \theta_{22}(t, t_f) \end{pmatrix} \begin{pmatrix} x^*(t_f) \\ p(t_f) \end{pmatrix}$$

where $\Theta(t, t_f)$ has been partitioned into 4 $n \times n$ blocks. Separating the two equations, and recalling that $p(t_f) = P_1 x^*(t_f)$, we have

$$x^*(t) = \theta_{11}(t, t_f)\ x^*(t_f) + \theta_{12}(t, t_f)\, P_1\, x^*(t_f) \tag{4.21}$$

$$p(t) = \theta_{21}(t, t_f)\ x^*(t_f) + \theta_{22}(t, t_f)\, P_1\, x^*(t_f) \tag{4.22}$$

Finding $x^*(t_f)$ from (4.21) and using it in (4.22), we get

$$p(t) = [\theta_{21}(t, t_f) + \theta_{22}(t, t_f)\, P_1]\ [\theta_{11}(t, t_f) + \theta_{12}(t, t_f)\, P_1]^{-1}\ x^*(t). \tag{4.23}$$

Note that the two brackets are independent of x_o and only depend on system matrices. This leads us to introduce

$$P(t) \stackrel{\triangle}{=} [\theta_{21}(t, t_f) + \theta_{22}(t, t_f)\, P_1]\ [\theta_{11}(t, t_f) + \theta_{12}(t, t_f)\, P_1]^{-1}, \tag{4.24}$$

which results in the following form for the optimal control

$$p(t) = P(t)\, x^*(t) \Rightarrow u^*(t) = -R^{-1}(t)\ B^T(t)\ P(t)\ x^*(t) \tag{4.25}$$

which is in the feedback form!!! and is independent of x_o. The last problem is actually finding this $P(t)$! For this, we go back and differentiate (4.23) (i.e, $p(t) = P(t)x^*(t)$), which yields

$$\dot{p}(t) = \dot{P}(t)x^*(t) + P(t)\dot{x}^*(t)$$

using the expressions for $\dot{p}$ and $\dot{x}^*$, from (4.19),

$$-Q(t)x^*(t) - A^T(t)\, p(t) = \dot{P}(t)x^*(t) + P(t)A(t)x^*(t) - P(t)B(t)R^{-1}(t)p(t)$$

and using $p(t) = P(t)x^*(t)$, we get $\forall\ t \in [t_o, t_f]$

$$\left\{\dot{P}(t) + P(t)A(t) + A^T(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t)\right\} x^*(t) = 0,$$

for all $x^*(.)$ that follow from all possible x_o's. Therefore, we must have $\forall\ t \in [t_o, t_f]$

$$\begin{cases} \dot{P}(t) + P(t)A(t) + A^T(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t) = 0, \\ P(t_f) = P_1 \end{cases} \tag{4.26}$$

where the final condition is due to $p(t_f) = P_1 x^*(t_f)$. Equation (4.26) is known as the 'matrix (differential) Riccati' equation.

We can summarize the solution of LQR as follows:

ALGORITHM:
- solve (4.26) 'backwards in time' for $P(t)$
- store the gain matrix $K(t) = -R^{-1}(t)B^T(t)P(t)$
- implement $u(t) = K(t)x(t)$ on line

It appears that the problem is solved! (But we cannot just leave 'good enough' alone, can we?) We can come up with a lot of nifty results. For example, define

$$\begin{cases} \mathcal{L}(t, t_f, x_o, u^*(.)) \stackrel{\triangle}{=} x^{*T}(t)P(t)\, x^*(t) \\ \mathcal{L}(t_f, t_f, x_o, u^*(.)) = x^{*T}(t_f)P_1\, x^*(t_f) \end{cases} \tag{4.27}$$

that is, a function that starts from t_f and evolved backward toward t_o, for a given optimal trajectory.

It can be shown (see homework problems) that the derivative of this function satisfies the following

$$\dot{\mathcal{L}}(t, t_f, x_o, u^*(.)) = -x^{*T}(t)Q(t)x^*(t) - u^{*T}(t)R(t)u^*(t)$$

which implies that $\mathcal{L}(t, t_f, x_o, u^*(.))$ is the value of J if only the portion from t to t_f was integrated. As a result, $J(t_o, t_f, x_o, u^*(.)) = \mathcal{L}(t_o, t_f, x_o, u^*(.))$ or

$$J_{min} = x^{*T}(t_o)P(t_o)x^*(t_o). \tag{4.28}$$

This last result is quite useful for calculating the total 'cost' associated for a given set of initial conditions. It also leads to a great deal of insight into this optimal control problem.

We will now look into the case where $t \to \infty$. First, since from now on we are interested in very large terminal time, it makes sense to make the following assumption:

Assumption 4.2. *For indefinite horizon problem,* $P_1 = 0$.

Next consider two different values for the final time:
1) $t_f = t_2$
2) $t_f = t_1$
where $t_2 > t_1$. It can be shown (see HW) that

$$J(t_o, t_1, x_o) \leq J(t_o, t_2, x_o). \tag{4.29}$$

Note that we have eliminated u from the arguments of J since in the optimal case, $u(.)$ is a function of x. Now let us use the following definition:

$$\begin{cases} P^1(t) : \textit{The solution to } (4.26) \textit{ with } t_f = t_1 \\ P^2(t) : \textit{The solution to } (4.26) \textit{ with } t_f = t_2 \end{cases} \tag{4.30}$$

Clearly, (4.28), (4.29) and (4.30) imply that

$$P^1(t_o) \leq P^2(t_o) \cdots \quad .$$

This relations holds regardless of the value of x_o or behavior of the system matrices, $A(t)$, etc. Therefore we come to this 'bottom line':

Remark 4.3. *The longer you integrate the Riccati equation, the larger the value of $P(t_o)$.*

Now, we are ready to tackle the steady state case. While there are some applications for periodic systems, these results are mostly used for the time-invariant case. The steady state case is sometimes called 'the infinite horizon' case.

4.2 The Steady State Riccati Equation (SSRE)

Since we will deal with time invariant systems (from now on, that is), we can use $t_o = 0$ with out loss of generality. Also, note that from now on, we will use the following assumption:

Assumption 4.4. *Matrices A,B, Q, and R are all constant matrices.*

The problem, therefore is the following:

$$\begin{cases} \dot{x}(t) = Ax + Bu \\ x(0) = x_o \end{cases} \tag{4.31}$$

where we have dropped (t) from x and u, for simplicity. The cost function to be minimized is

$$J(x_o) = \int_0^\infty \{\, x^T Q x + u^T R u \,\}\, dt. \tag{4.32}$$

From the development of previous Section, we know that the optimal solution has the following form for the control law

$$u_{opt}(t) = -R^{-1}\, B^T\, P(t)\, x(t) \tag{4.33}$$

where $P(t)$ is obtained from

$$\begin{cases} \dot{P}(t) + P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t) + Q = 0, \quad \forall\ t \in [t, \infty] \\ P(\infty) = 0. \end{cases} \tag{4.34}$$

By the discussion at the end of the previous Subsection, it is clear that as (4.34) is integrated backwards

$$P(t_2) \leq P(t_1) \quad if \quad t_2 \geq t_1$$

(because the duration of integration for $P(t_1)$ is longer). The question becomes: "As we integrate more and more, does P (a nondecreasing function) blow up or does it converge to something?" Note that by (4.28), $P(0)$ have to be at least semi-positive definite.

Next, suppose that (A, B) is stabilizable; i.e., $\exists$ a gain matrix such that $(A - BK)$ is stable. Now use this gain for control; i.e., use $u = -Kx$, which results in a closed loop system of $\dot{x} = (A - BK)x$. Clearly, this may not be the optimal control. Let us calculate the cost functional J for this control law

$$J = \int_0^\infty \{\, x^T Q x + u^T K^T R K x \,\}\, dt = \int_0^\infty \{\, x^T (Q + K^T R K) x \,\}\, dt.$$

However, for this system we know that $x(t)$ has the from

$$x(t) = e^{(A-BK)t}x_o$$

which results in

$$J = \int_0^\infty \{\, x_o^T\, e^{(A-BK)^T t}\, (Q + K^T R K)\, e^{(A-BK)t}\, x_o \,\}\, dt.$$

From basic definitions of stability and exponential stability, it follows that this J is finite. Since this is not necessarily the optimal J, the optimal J is finite, as well. By (4.28), it follows that P does not blow up! We make the following observation:

Remark 4.5. *Stabilizability implies the existence of an optimal control law. Indeed, it also produces upper bounds for J and, hence, $P(0)$. Therefore, it also implies the convergence of $P(t)$.*

Once this convergence occurs, say $P(t) \to P$, the derivative in (4.34) disappears and P satisfies the following equation (known as the 'Algebraic Riccati Equation or ARE')

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \quad ARE \tag{4.35}$$

The convergence can be shown "formally", by assuming $P(t)$ converges to some $\overline{P}$ (due to convergence of J). Then it can be shown that this is the same as P in (4.35), for any positive semi definite solution of (4.35).

Remark 4.6. *Uniqueness of the positive semi-definite solution to* (4.35) *- and stability of the closed loop sysem - is established through the use of another assumption: (A, Q) observable.*

4.3 The Potter's Method

Let us start with the optimal case

$$\dot{x}(t) = Ax + Bu = (A - BR^{-1}B^T P)\, x$$

and for brevity, define

$$G \stackrel{\triangle}{=} BR^{-1}B^T \tag{4.36}$$

so that the closed loop system and ARE can re written as

$$\begin{cases} \dot{x}(t) = (A - GP)x \\ PA + A^T P - PGP + Q = 0. \end{cases} \tag{4.37}$$

Matrix $(A - GP)$ is the closed loop matrix. Its eigenvalues and eigenvectors are called the closed loop eigenvalues and eigenvector, respectively. Denote the eigenvalues and eigenvectors of the closed loop system by

$$(A - GP)X_i = \lambda_i X_i. \tag{4.38}$$

From (4.37), we have $P(A - GP) = -A^T P - Q$, therefore (using (4.38))

$$P(A - GP)X_i = \lambda_i P X_i = -A^T P X_i - Q X_i. \tag{4.39}$$

Now, putting (4.38) and (4.39) in matrix form

$$\begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix} \begin{Bmatrix} X_i \\ PX_i \end{Bmatrix} = \lambda_i \begin{Bmatrix} X_i \\ PX_i \end{Bmatrix}. \tag{4.40}$$

The $2x \times 2x$ matrix in (4.40) is called the 'Hamiltonian' matrix. From now on, we will call it H ; i.e.,

$$H \stackrel{\triangle}{=} \begin{bmatrix} A & -G \\ -Q & -A^T \end{bmatrix}. \tag{4.41}$$

From (4.40), we see that every closed loop eigenvalue is an eigenvalue of H. Also, note the relationship between the top and bottom halves of the eigenvectors of H corresponding to these eigenvalues. If everything goes o.k., we will be looking for stable eigenvalues of H (better have stable closed loop!) and exploit the structure of these eigenvectors. Note that if we call the bottom of each stable eigenvector $Y_i = PX_i$, then by stacking these n vector equations next to one another, we have

$$[\, Y_1 \; Y_2 \; \ldots Y_n \,] = P \, [\, X_1 \; X_2 \; \ldots X_n \,].$$

That is how Potter solved the problem about 30 years ago. Consider this algorithm:

- Form H in (4.41) (assume $Re(\lambda(H)) \neq 0$)
- Solve for eigenvalues and eigenvectors. Choose the stable ones only.
- Stack the stable eigenvectors next to one another to form a $2x \times n$ matrix. Call the top half X and the bottom half Y.
- Find P according to $P = Y\,X^{-1}$

It is easy to show (see HW, also it is one of the main properties of Hamiltonian type matrices) that if λ is an eigenvalue of H, so is $-\lambda$ (i.e., eigenvalues are symmetric with respect to the imaginary axis). That is, there cannot be more that n stable (or unstable) eigenvalues for H!

What are the pitfalls? What if some of the eigenvalues of H have zero real parts? What if X is not invertible? We will deal with these through homework problems and class discussion.

Remark 4.7. *There has been a great deal of research on the minimum (i.e., sufficient and necessary) conditions needed for the existence of a $P \geq 0$ that stabilizes the closed loop (ans: detectability and stabilizability), those needed to have $P > 0$, efficient numerical algorithms (e.g., Schur methods), etc. In class discussions, we will address some of these issues and talk about references for most others.*

4.4 PROBLEM SET

P1. When we write x^TQx, or similar expressions, we typically assume Q is symmetric. Show that this is not an important -or restrictive - assumption, as long as Q is used in these quadratic forms only.

P2. Let $X(\epsilon) = A + \epsilon B + \epsilon^2 C$, with $A > 0$, $C \geq 0$ and ϵ scalar (possibly negative). Also, A, B, C are independent of ϵ. Show that if A is the minimum value of $X(\epsilon)$, for all ϵ, then B must be zero.

P3. Show that

$$\int_{t_o}^{t_1} \int_{t_o}^{\tau} A(t,\tau)\, dt\, d\tau = \int_{t_o}^{t_1} \int_{t}^{t_1} A(t,\tau)\, d\tau\, dt$$

P4. Let $\Phi(t,\tau)$ be the state transition matrix for $A(t)$. Show that

$$\frac{d}{dt}\Phi(\tau,t) = -\Phi(\tau,t)A(t)$$

P5. If $p(t)$ is defined by (4.16), use Leibniz rule to show that (4.18) is true.

P6. Show that in (4.26), the solution P is symmetric; i.e., $P(t) = P^T(t)$

P7. calculate and simplify the derivative of $\mathcal{L}$ defined in (4.27). Show (4.28) holds (hint: consider $\hat{J} = \int_t^{t_f} x^TQx \cdots$)

P8. Show that (4.29) holds (remember to set P_1 to zero).

P9. Show that the eigenvalues of H in (4.41) are symmetric with respect to the imaginary axis. (hint: one way is to lok into the possibility of having $(-X^TP \quad X^T)^T$ as eigenvector of H^T)

P10. Show that if (A, B) is controllable and (A, Q) is observable, then eigenvalues of H will not have zero real parts.

P11. Using (4.35), and not the Hamiltonian, show that

$$Q > 0 \Rightarrow P > 0 \;\; \textit{and/or} \;\; (A,Q)\; \textit{observable} \quad \Rightarrow P > 0$$

(think of J). What about

$$Q > 0 \Rightarrow (A - GP)\; \textit{stable}$$

$$Q \geq 0, \quad (A,Q)\; \textit{observable} \Rightarrow (A - GP)\; \textit{stable}$$

Can you use the basic definition of J to interpret these last results?

5 EXTENSIONS TO LQR

We start with the *infinite horizon, time invariant* problem, which has been discussed in some details. One potential problem may be that while the LQR problem we discussed minimizes the cost functional, it may not result in good response (e.g., the state vector dies too slowly) or does not fit our goals (our objectives do not fit the form of J we have been using). In this Section, we will try to deal with a few of these issues. Unless specifically needed, we will simplify notations by using x, y, etc. instead of $x(t)$, $y(t)$, etc. The time dependency will be clear from the context.

5.1 Cross Terms in the Cost Functional

Consider the same system as before, i.e,

$$\begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_o. \end{cases} \tag{5.1}$$

Instead of the J we used before, however, let us try to minimize the following cost functional

$$J(x_o) = \int_0^\infty \{\, x^T Q x + u^T R u + 2x^T S u \,\}\, dt. \tag{5.2}$$

where S is a matrix of appropriate dimension.

Remark 5.1. *Cost functionals of the form* (5.2) *are encountered when, for example, one is interested in minimizing the integral of control effort (i.e., the* $u^T R u$ *term) plus the square of the norm of some outputs* $y = Cx + Du$*. The term* $\|y\|^2$ *would, typically, have cross terms.*

The first step in solving this problem is to do the following manipulations (of the 'completing the square' variety);

$$u^T R u + 2x^T S u + x^T Q x = (u + R^{-1} S^T x)^T\, R\, (u + R^{-1} S^T x) + x^T (Q - S R^{-1} S^T) x. \tag{5.3}$$

Next, define the following modified dynamics

$$\begin{cases} \dot{x} = (A - B R^{-1} S^T) x + B\tilde{u} \\ \tilde{u} = u + R^{-1} S^T x. \end{cases} \tag{5.4}$$

The cost funcctional in (5.2) can now be written as

$$J(x_o) = \int_0^\infty \{\, x^T (Q - S R^{-1} S^T) x + \tilde{u}^T R \tilde{u} \,\}\, dt. \tag{5.5}$$

Note that (5.4) and (5.5) form a standard LQR problem, i.e., one obtains the positive definite solution of

$$P(A - B R^{-1} S^T) + (A - B R^{-1} S^T)^T P - P B R^{-1} B^T P + Q - S R^{-1} S^T = 0$$

and implements the control $\tilde{u} = -R^{-1}B^T Px$ in (5.4) to minimizes (5.2) or (5.5). Implementing this control law in (5.4), however, is equivalent to using the following control law in our original sysem (i.e., (5.1))

$$u(t) = \tilde{u}(t) - R^{-1}S^T x(t) = -R^{-1}(B^T P + S^T)x(t). \tag{5.6}$$

Finally, for the problem to work, we need to make the following assumptions:

Assumption 5.2. *Matrix S is chosen (e.g., small enough) so that $\tilde{Q} = Q - SR^{-1}S^T \geq 0$.*

Assumption 5.3. *The pair $[(A - BR^{-1}S^T)\ ,\ \tilde{Q}]$, is observable and the pair $[A, B]$ is controllable.*

Remark 5.4. *The first assumption above is needed for the cost functional* (5.5) *to make sense. The second one is needed for technical (yet important) reasons, such as stability of the closed loop. Both can be met by having S small enough, if $Q > 0$. Also note that we do not need to require $[(A - BR^{-1}S^T), B]$ to be controllable, if $[A, B]$ was controllable to start with.*

5.2 Regulators with a Prescribed Degree of Stability - α shifts

Recall that in the infinite horizon problem, we traded the ability to set the terminal time with the ability to solve the ARE and, hence, come up a much easier controller (both to calculate and to implement). However, one may ask: what if the resulting problem reduced x at unacceptably slow rates? In this subsection, we deal with the case where we wish to kill the state faster than the standard LQR. To do this, we modify the cost functional. Again we start with the dynamics

$$\begin{cases} \dot{x}(t) = Ax + Bu \\ x(0) = x_o \end{cases} \tag{5.7}$$

but try to minimize

$$J(x_o) = \int_0^\infty \{\, e^{2\alpha t}(x^T Q x + u^T R u)\,\}\, dt, \qquad \alpha \geq 0. \tag{5.8}$$

Note that this cost functional will try to force x to die at least as fast as $e^{-\alpha t}$ (why?) The constant α, therefore, can be used to force the controller to act faster! The problem is that (5.8) creates a time varying LQR problem, which destroys a great deal of convenience. The question is: Can we trick the controller so that is solves (5.7) and (5.8) by trying a related time invariant LQR problem? The answer, as you may have guessed, is yes!

First, we need to define

$$\begin{cases} \hat{x}(t) = e^{\alpha t}\, x(t) \\ \hat{u}(t) = e^{\alpha t}\, u(t). \end{cases} \tag{5.9}$$

With these definitions, the cost functional in (5.8) can be written as

$$J(x_o) = \int_0^\infty \{\, (\hat{x}^T Q \hat{x} + \hat{u}^T R \hat{u})\,\}\, dt \triangleq \hat{J}(x_o). \tag{5.10}$$

Now, taking the derivative of $\hat{x}$ in (5.9), we obtain

$$\dot{\hat{x}}(t) = \alpha e^{\alpha t} x(t) + e^{\alpha t}\dot{x}(t) = \alpha \hat{x}(t) + A\hat{x}(t) + B\hat{u}(t)$$

or

$$\begin{cases} \dot{\hat{x}}(t) = [A + \alpha I]\hat{x}(t) + B\hat{u}(t) = \hat{A}\hat{x}(t) + B\hat{u}(t) \\ \hat{x}(0) = x_o \end{cases} \tag{5.11}$$

Note that (5.11) and (5.7) are equivalent; i.e, one implies another, as long as we used the definitions in (5.9). Indeed, it is an easy exercise to show that a given $\hat{u}$ in (5.11) results in a $\hat{x}$ which is exactly $e^{\alpha t}$ times the x that results from implementing $u = e^{-\alpha t}\hat{u}$ in (5.7)!

As a result, our original problem is tranformed into minimizing (5.10), subject to (5.11), which looks like a standard time invariant LQR problem. Therefore, we solve for the positive definite solution of

$$P_\alpha(A+\alpha I)+(A+\alpha I)^T P_\alpha - P_\alpha B R^{-1} B^T P_\alpha + Q = 0 \tag{5.12}$$

where the subscript α is used to underline the fact that P depends on α. Next, we can write the the optimal control law

$$\hat{u}(t) = -R^{-1}B^T P_\alpha \hat{x}(t). \tag{5.13}$$

The last step is to find the control law for the actual system (i.e., (5.7)). For this, simply note that

$$u(t) = e^{-\alpha t}\hat{u}(t) = -e^{\alpha t}R^{-1}B^T P_\alpha \hat{x}(t) = -R^{-1}B^T P_\alpha x(t). \tag{5.14}$$

Remark 5.5. *The only change from the standard case is that the ARE has been changed (hence, P_α). Since it is easy to show that the controllability and observability are not affected by replacing A with $(A+\alpha I)$, controllability and observability of the original system implies the same properties for the system in* (5.10) *and* (5.11)*. Note, however, that this is not necessarily true for stabilizability and detectability.*

Remark 5.6. *This method is often called 'the α shift' for obvious reasons. You can say that by using $A+\alpha I$, we are pretending our system to be a lot more unstable (or less stable). As a result, the control will 'work harder' to push everything further back to the left half plane, since it tries to find a K such that the eigenvalues of $A+BK+\alpha I$ are in the left half plane. This imples that the same K would result in closed loop (i.e., $A+BK$) eigenvalues with real parts less than $-\alpha$.*

5.3 The Servo and Tracking Problems

Consider the follwoing system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{5.15}$$

where y is the measured output of the system. The servo and tracking problems concern the issue of following a trajectory $\tilde{y}(t)$, by minimizing the following cost functional

$$J(x_o) = \int_0^T \{ (y - \tilde{y})^T Q (y - \tilde{y}) + u^T R u \} \, dt. \tag{5.16}$$

where the composition of J is motivated by the need to reduce error between desired trajectory and output, while having some flexibility with respect to control effort. We will use the terminal time T so that finite duration control can also be attempted.

The desired trajectory, $\tilde{y}$, may be constant or time varying. The treatment here deals with both. For the sake of simplicity, however, we have dropped the explicit dependence on t. A typical approach is to first generalize this cost functional into

$$J(x_o) = \int_0^T \{ (y - \tilde{y})^T Q_2 (y - \tilde{y}) + \bar{y}^T Q_1 \bar{y} + u^T R u \} \, dt, \quad Q \geq 0, Q_1 \geq 0, \tag{5.17}$$

where

$$\begin{cases} \bar{y} = \bar{C} x \\ \bar{C} = I - LC = I - C^T (CC^T)^{-1} \, C. \end{cases} \tag{5.18}$$

Remark 5.7. *The generalization in* (5.17) *can be made without any loss of generality since we can always set* $Q_1 = 0$.

Based on the definitions in (5.18), we have

$$\begin{cases} CL = I, \quad C\bar{y} = 0 \\ x = \bar{y} + x_1 \quad , \quad x_1 \in Range(C^T). \end{cases} \tag{5.19}$$

Remark 5.8. *Considering the basic definitions of orthogonal projections, it is simple to see that* $\bar{C}$ *is the (orthogonal) projection matrix onto the othogonal complement of range space of* C^T*; i.e, the null space of* C*. In other word,* $\bar{y}$ *is part of the state vector that is not seen by* $y = Cx$.

Next, we define

$$\begin{cases} \tilde{x} = L\tilde{y} \\ which \ \Rightarrow \tilde{y} = C\tilde{x}. \end{cases} \tag{5.20}$$

Note that by (5.20), we have 'found' a state trajectory that results in our desired output trajectory, if passed through C. As a result, we can turn the output error based cost functional of (5.16) or (5.17) with the following, state tracking error based, cost functional

$$\begin{aligned} J(x_o) &= \int_0^T \{\, (x-\tilde{x})^T [C^T Q_2 C + \bar{C}^T Q_1 \bar{C}](x-\tilde{x}) + u^T R u \,\}\, dt \\ &= \int_0^T \{\, (x-\tilde{x})^T Q (x-\tilde{x}) + u^T R u \,\}\, dt. \end{aligned} \tag{5.21}$$

As a result, from now on, we can (without any loss of generality) focus on problems that have state trajectory error terms in their cost functionals.

5.3.1 The Servo Problem

Let us assume that trajectory is (or could be!) from the following model

$$\left\{ \begin{array}{l} \dot{z} = Fz \\ \tilde{y} = Hz\ , \quad (F,H)\ \ observable \end{array} \right. \tag{5.22}$$

for some H and F. Note that z plays the same role as $\tilde{x}$ in (5.20). The objective here is to minimize costs of the form (5.21), with $\tilde{x}, L, etc.$ as discussed previously.

Remark 5.9. *Equation* (5.22) *can be used to model a great variety of output trajectories. For example: every polynomial function of time (of any order).*

We can distinguish two cases:

Case 1 z is directly available:

The original system and the model in (5.22) - as well as the cost functional - can be written in terms of the following augmented system

$$\left\{ \begin{array}{l} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ \hat{J} = \int_0^T \{\, \hat{x}^T \hat{Q} \hat{x} + u^T R u \,\}\, dt, \end{array} \right. \tag{5.23}$$

where we have used

$$\hat{A} = \left(\begin{array}{cc} A & 0 \\ 0 & F \end{array} \right) \ , \ \hat{B} = \left(\begin{array}{c} B \\ 0 \end{array} \right) \ , \ \hat{Q} = \left(\begin{array}{cc} Q & -QLH \\ -H^T L^T Q & H^T L^T Q L H \end{array} \right). \tag{5.24}$$

From the basic LQR, we know the optimal control law is

$$u^*(t) = -R^{-1} \hat{B}^T \hat{P}(t)\ \hat{x}(t) \tag{5.25}$$

where $\hat{P}(t)$ is the solution to the Riccati matrix differential equation

$$\begin{cases} -\dot{\hat{P}}(t) = \hat{A}^T\hat{P}(t) + \hat{P}(t)\hat{A} - \hat{P}(t)\hat{B}^T R^{-1}\hat{B}\hat{P}(t) + \hat{Q} \\ \hat{P}(T) = 0. \end{cases} \tag{5.26}$$

To simplify the controller, let us partition $\hat{P}(t)$

$$\hat{P}(t) = \begin{bmatrix} P(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{bmatrix}. \tag{5.27}$$

We can write the control law of (5.25) as

$$\begin{cases} u^*(t) = K_1(t)x(t) + K_2(t)z(t) \\ K_1(t) = -R^{-1}B^T P(t) \\ K_2(t) = -R^{-1}B^T P_{12}(t) \end{cases} \tag{5.28}$$

with

$$\begin{cases} -\dot{P}(t) = P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t) + Q \\ -\dot{P}_{12}(t) = P_{12}(t)F + A^T P_{12}(t) - P(t)BR^{-1}B^T P_{12}(t) - QLH \\ -\dot{P}_{22}(t) = P_{22}(t)F + F^T P_{22}(t) - P_{12}(t)BR^{-1}B^T P_{12}(t) + H^T L^T QLH \\ P(T) = P_{12}(T) = P_{22}(T) = 0. \end{cases} \tag{5.29}$$

Similarly, it can be shown that the final cost wil be

$$J^* = x_o^T P(0)x_o + 2x_o^T P_{12}(0)z(0) + z^T(0)P_{22}(0)z(0). \tag{5.30}$$

Remark 5.10. *The initial time of zero was used for simplicity. If needed, simply replace the appropriate t_o in the usual places (e.g., in the cost functionals and in* (5.30)*). Similarly, we have used no terminal time penalty. This can be added easily. Start with a P_1 to penalize the error in the state tracking at time T, find the corresponding $\hat{P}_1$ for* (5.23)-(5.25)*. It will only affect, directly, the final conditions in* (5.29)*.*

Remark 5.11. *Since the duration is finite, the control is time varying. However, by inspecting* (5.28)*, you can see that the control has two parts. One is the exact same term you would get out of standard LQR (i.e., if you were interested in killing the state and not tracking). The secon part was only dependent on P_{12} and z. In effect, this is a feedforward type of compensation! Lastly, note that the equation for P_{22} need not be integrated for control, since it is not used in either of the other two equations. It is only used to evaluate the total cost in* (5.30)*.*

Case 2 z is not available directly:

We simply desgin an observer for the model (5.22) !! The rest is identical (but much messier!)

5.3.2 The Tracking Problem

In the last subsection, we discussed the servo problem, in which we assumed the existence of a model (or another system) which is the sourse of the desired trajectory (i.e., (5.22)). As a result, we require the knowledge of the model (i.e., F and H) and on-line measurement of z (the desired state trajectory). Now, let us consider the case where the knowledge of F and H is not possible, but we may know the complete trajectory history (i.e., we know $\tilde{y}(t)$ for all $t \in [0, T]$).

By remark 5.11, we know that the optimal control has two parts. The part that we will have problems with is the feedforward term, since its evaluation required $P_{12}(t)$ which itself needed H and F, see (5.29). So we will start with the feedformard term

$$u_{ff}(t) = -R^{-1}B^T P_{12}(t)z(t) = -R^{-1}B^T b(t)$$

where we have defined $b = P_{12}z$. Let us take derivative of this vector. After some minor manipulations, we have

$$-\dot{b}(t) = (A - BR^{-1}B^T P(t))^T\, b(t) - QL\tilde{y}(t) \quad ,\ b(T) = P_{12}(T)z(T) = 0, \tag{5.31}$$

where equations (5.22) and (5.29) are used.

The tracking problem is then solved by integrating (5.31) backward in time from T to zero and implementing the follwoing control law

$$u^*(t) = -R^{-1}B^T[P(t)x(t) + b(t)].$$

Remark 5.12. *If we let $T \to \infty$, certain simplifications can be made. For example, the backward integration in* (5.31) *will be well behaved (since $(A - BR^{-1}B^T P)$ is the closed loop LQR matrix and stable). Also, if $\tilde{y}$ is constant, then $b(t)$ actually converges to a constant (think of the integration in* (5.31) *as the response of a stable system to constant input) and the feedforward term becomes a constant, as well (an offset!).*

5.4 PROBLEM SET

Exercise 5.13. *Show that if* (A, B) *is controllable, so is* $[(A - BR^{-1}S^T), B]$.

Exercise 5.14. *Show the details of the tranformation of* (5.2) *to* (5.5).

Exercise 5.15. *Show that controllability and observability are not affected by replacing* A *with* $A + \alpha I$. *What about detectability and stabilizability?*

Exercise 5.16. *What is the rate of decay for* $x(t)$ *if* (5.14) *is used? Show it by (1) using* $\hat{x}$ *and (2) Lyapunov arguments based on behavior of* $x^T P_\alpha x$ *where* P_α *is the solution of* (5.12).

Exercise 5.17. *What is the value of* J_α *in terms of* P_α*? Is* $J_\alpha \geq J_{\alpha=0}$ *if* $\alpha \geq 0$*? Hint: Take* $\frac{d}{d\alpha}$ *of* (5.12) *and show* $\frac{dP_\alpha}{d\alpha} > 0$ *through a Lyapunov argument. Then consider* $\frac{d}{d\alpha} J_\alpha$.

Exercise 5.18. *Consider* (5.18). *Show that* LC *is the orthogonal projection operator onto the range of* C^T *and, hence,* $I - LC$ *is the projection onto the null spave of* C.

Exercise 5.19. *Again, in* (5.18), *show* $(LC)(LC) = LC$, $(I - LC)LC = 0$.

Exercise 5.20. *Show* (5.17) *and* (5.21) *are the same.*

Exercise 5.21. *Verify* remark 5.9. *Hint: Consider an* F *with all entries zero with ones on the super diagonal.*

Exercise 5.22. *Verify* (5.23),(5.24) *and* (5.28)-(5.30).

Exercise 5.23. *What happens if there is a terminal penalty term for the servo problem. Work the details. How does it effect the tracking problem.*

Exercise 5.24. *Work the details of the* remark 5.12.

6 OUTPUT FEEDBACK DESIGN

When the whole sate vector is not available for feedback, i.e, we can measure only

$$y = Cx.$$

6.1 Review of observer design

Recall from the first class in linear systems that a simple control law would be

$$u = Kx \Longrightarrow \dot{x} = (A + BK)x$$

where K is chosen so that $A + BK$ is stable (from pole placement of LQR, etc). Now if you can not measure x, then you use an output feedback design. Static output feedback design; i.e,. $u = Ky$ turns out to be relatively hard to solve (unless you do trial and error) - more on this later. The most common - and systematic approach is to use a dynamic output feedback, where the controller (or compensator) has its own dynamics (recall the typical compensator box from classical control course). The simplest form is an observer structure; i.e., use $u = K\hat{x}$ where $\hat{x}$ is an estimate for the actual x and comes from a copy of the model we construct with our control hardware (or software)

$$\dot{x} = Ax + Bu \tag{6.1}$$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - C\hat{x}) \tag{6.2}$$

$$u = K\hat{x} \tag{6.3}$$

The trick, in this simple approach, is to pick a good L such that $\hat{x} \longrightarrow x$ relatively soon. First, let us write the model in terms of x and $e = x - \hat{x}$

$$\dot{x} = Ax + BK\hat{x} = (A + BK)x - BKe \tag{6.4}$$

$$\dot{e} = (A + LC)e \tag{6.5}$$

$$\tag{6.6}$$

which can be written as the following for $x_{cl}^T = (x^T \quad e^T)$

$$\dot{x}_{cl} = A_c x_{cl} \quad , \quad A_{cl} = \begin{bmatrix} A + BK & -BK \\ 0 & A + LC \end{bmatrix}$$

since $del(A_{cl}) = det(A + BK)\ .\ det(A + LC)$, we have closed loop stability (i.e. $e(t) \longrightarrow 0$ as time gets larger, which means $\hat{x} \longrightarrow x$) as long as $A + LC$ stable. Note that this is independent of the choice of K - a trivial case of 'separation principle,' Also note that the poles of $A + LC$ set how fast $e(t)$ dies. It is common to use a role of thumb that days the least stable pole of $A + LC$ should be three times as fast as the dominant modes of $A + BK$.

6.2 The Kalman filter and the LQG

In this subsection, we will review -very briefly- Kalman filter equations for the Linear Quadratic Guassian problem (LQG). Due to the time limitation, this review will be extremely brief.

Consider the following stochastic linear system

$$\begin{cases} \dot{x} = Ax + Bu + \Gamma\zeta(t) \\ y = Cx + \eta(t) \end{cases} \tag{6.7}$$

where $\zeta(t)$ and $\eta(t)$ are vector random processes (i.e., the process noise and measurement noise, respectively).

After ignoring a great deal of effort (and potential pitfalls) with respect to the well-posedness of (6.7), we assign

$$\begin{cases} \mathcal{E}\left[\zeta(t)\right] = \mathcal{E}\left[\eta(t)\right] = 0 \quad (zero\ mean) \\ \mathcal{E}\left[\zeta(t)\eta^T(\tau)\right] = 0 \quad (uncorrelated) \\ \mathcal{E}\left[\zeta(t)\zeta^T(\tau)\right] = Q_o\ \delta(t-\tau)\ ,\ Q_o \geq 0 \\ \mathcal{E}\left[\eta(t)\eta^T(\tau)\right] = R_o\ \delta(t-\tau)\ ,\ R_o > 0 \end{cases} \tag{6.8}$$

where $\mathcal{E}$ is the expectation operator (think ensemble avarage, or in the case of ergotic signals, time averages). Relying only on the measurement $y(t)$, we wish to 'estimate' the $x(t)$ - and denote the estimate by $\hat{x}$ - such that the following error is minimized:

$$\begin{cases} e(t) = x(t) - \hat{x}(t) \\ minimize \quad \mathcal{E}\left[e^T(t)e(t)\right] = min\ \mathcal{E}\left[\|x(t) - \hat{x}(t)\|^2\right]. \end{cases} \tag{6.9}$$

After a great deal of work (roughly two quarters of stochastic processes worth!) the following is obtained for the *steady state* case:

Observer Equation :

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_f[y(t) - C\hat{x}(t)] \tag{6.10}$$

$$K_f = SC^T R_o^{-1} \tag{6.11}$$

$$AS + SA^T - SC^T R_o^{-1} CS + \Gamma Q_o \Gamma^T = 0. \tag{6.12}$$

Note that with this obsever, the error equation becomes

$$\dot{e}(t) = [A - K_f C]\, e(t) + [\Gamma \quad -K_f] \begin{bmatrix} \zeta(t) \\ \eta(t) \end{bmatrix}. \tag{6.13}$$

After noting the similarity of these equations to those of the LQR method, we have the following :

Remark 6.1. *The Riccati equation in* (6.12) *resembles the one encountered in LQR. Notice the duality between the two, by replacing* B *with* C^T *and* A *by* A^T*. As a result, a great many of the results and techniques we discussed earlier apply here, as well. For example: if* (A, C) *is observable and* (A, Γ) *is controllable, then* (6.12) *has a unique solution,* $S > 0$*, and* $(A - K_f C)$ *is stable.*

Remark 6.2. *In* (6.10) *if the noise terms are ignored, then we have an observer which is obtained from* (6.11)-(6.12)*, and results in a stable closed loop. Further, if the model has noise (as in* (6.7)*) then this observer minimizes* (6.9)*, as well. This is the approach we will choose; i.e., we will use* (6.12) *and* (6.11) *to desgin 'desirable' observers (rather than pole placement methods, for example). Lastly, matrices* Q_o *and* R_o *can be interpreted as the intensity of the process noise and measurement noise, respectively. A very large* R_o*, for example, denotes high levels of measurement noise. One might expect that such a system would end up with small observer gains (why?), which is indeed true. What is the dual problem in LQR?*

6.3 The Linear Quadratic Guassian Compensator - LQG

Consider the dynamical system in (6.7), subject to (6.8). The LQG problem is to design a controller of the form

$$u(t) = f[\ y(\tau)\ ,\ \tau \leq t\ ] \tag{6.14}$$

to minimize

$$J = \mathcal{E}[\lim_{T\to\infty} \frac{1}{T} \int_o^T (\ x^T H^T H x + u^T R u\ )\, dt] \tag{6.15}$$

under the following assumptions

$$\begin{cases} (A, B), \ \ and \ \ (A\Gamma)\ controllable(stabilizable) \\ (A, C) \ \ and \ \ (A, H)\ observable \\ R_o > 0 \ \ , \ \ R > 0 \ \ Q_o \geq 0. \end{cases} \tag{6.16}$$

The solution to this problem is the following:

$$\begin{cases} u(t) = -K_c \hat{x}(t) \\ K_c = R^{-1} B^T P \\ PA + A^T P - PBR^{-1}B^T P + H^T H = 0 \\ \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_f[y(t) - C\hat{x}(t)] \\ K_f = SC^T R_o^{-1} \\ AS + SA^T - SC^T R_o^{-1} CS + \Gamma Q_o \Gamma^T = 0. \end{cases} \tag{6.17}$$

Remark 6.3. *Equation* (6.17) *imples that the optimal solution can be separated into full state controller and observer design. This principle of separation in stochastic control works similar to the one encountered in pole-placement type controllers (in deterministic setting). Its proof, however, is quite complicated. Note the control consists of a LQR step plus an observer step (which is the steady state Kalman filter).*

6.4 Problem Set

Exercise 6.4. *Ignoring the noise (i.e., η and ζ), write the combined closed-loop state space form (in terms of state varaibles x and $\hat{x}$).*

Exercise 6.5. *Define the error to be $x - \hat{x}$. Write the combined closed-loop state state form in terms of state variables x and e. Are the eigenvalues of the closed loop system the same as in the previous exercise? Why?*

Exercise 6.6. *Again, set the noise to be zero. What is the transfer function of the compensator? That is, if we write the control as in $u(s) = H(s)\, y(s)$, what is $H(s)$, where '$x(s)$' denotes the Laplace Transform of $x(t)$. Try to draw the block diagram of this problem.*

Exercise 6.7. *Is the compensator (i.e., $H(s)$) stable? What are some of the possible problems with unstable compensators?*

7 MIMO ZEROS

We are interested in MIMO zeros for several reasons. Similar to the SISO case, nonminimum-phase zeros will effect the performance (degradation) of the control system. As we will see, the asymptotic behavior of the Riccati equation depends on the place of the zeros of certain transfer functions. Such results are somewhat similar to the SISO root-locus arguments on the poles as the gain becomes large (and the poles go to the open loop zeros).

Nonminimum phase zeros are, unfortunately, common in real life. A common example is a flexible beam with non-colocated actuator and sensor.

7.1 Transfer Function Approach

We start by going back to the single-input single-output (SISO) case. Consider the transfer function

$$G(s) = \frac{s+2}{(s+3)(s+4)}.$$

Now choose the input to be of the form $u(t) = u_o\, e^{-2t}$. It is simple to show that this input results in an output of the form $y(t) = u_o e^{-3t} - u_o e^{-4t}$; i.e., the output does not contain any terms of the form e^{-2t}. If the input had a term e^{-6t}, for example, the output *would* have the term e^{-6t} in it. Indeed, this can be generalized to second order terms (complex roots). Due to this property, we can think of the above development as if the system had 'blocked' the transmission of the signal of the form e^{-2t}.

This approach can be generalized to multi-input multi-outputs (MIMO) system by using the following for the input

$$u(t) = \underline{u}_k\, e^{z_k t}$$

where $\underline{u}_k$ and z_k are possibly complex vector and scalar, respectively. Now the question becomes: Are there any *pairs* $(\underline{u}_k, z_k)$ such that the output of a MIMO plant does not contain a $e^{z_k t}$ term?

Example 7.1. *Consider the plant*

$$y(s) = G(s)u(s) = \begin{bmatrix} \frac{s+1}{s^2} & 0 \\ 0 & \frac{s+2}{s^2} \end{bmatrix} u(s)$$

where it appears that $s = -1$ and $s = -2$ are zeros of the system. Indeed, it is easy to show that

$$u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \Rightarrow y(t) \text{ without an } e^{-t} \text{ term}$$

and

$$u(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \Rightarrow y(t) \text{ without an } e^{-2t} \text{ term.}$$

Definition 7.2. *A system, $G(s)$, has a transmission zero at $s = z_k$ if there exist vector u_k and scalar z_k such that $u(t) = u_k\, e^{z_k t}$ results in the output $y(t)$ not containing $e^{z_k t}$ terms. The vector u_k is called the (zero) direction of z_k.*

Using slightly different definitions, different kinds of zeros have been introduced, such as 'blocking zero' or 'transmission zero' or 'transmission blocking zeros' etc. If $G(s)$ is minimal (i.e., controllable and observable) then there is only one kind of zero, whatever it is called.

Definition 7.3. *A unimodular matrix is a square polynomial matrix (i.e., a matrix with entries polynomial powers of s), where the determinant is a nonzero constant. The inverse of such a matrix always exists and it is unimodular itself.*

7.2 Calculating Zeros

The basic idea is to reduce the problem into finding $G_1(s)$, $M(s)$, and $N(s)$, such that

$$G_1(s) = M(s)\; G(s)\; N(s) = \begin{bmatrix} \frac{\epsilon_1(s)}{\psi_1(s)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_2(s)}{\psi_2(s)} & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & \frac{\epsilon_p(s)}{\psi_p(s)} \end{bmatrix} \tag{7.1}$$

where $M(s)$ and $N(s)$ are unimodular matrices, and for each i, polynomials $\epsilon_i(s)$ and $\psi_i(s)$ are relatively prime (i.e., no common roots) with the additional property that

$$\epsilon_1(s) \mid \epsilon_2(s) \mid \epsilon_3(s) \mid \ \ldots \ \mid \epsilon_p(s) \tag{7.2}$$

and

$$\psi_p(s) \mid \psi_{p-1}(s) \mid \ \ldots \ \mid \psi_1(s) \tag{7.3}$$

where $A|B$ denotes 'A divides B', without a remainder. Now we can say: The special form of $G_1(s)$ is called the 'Smith McMillan Form'.

$$Transmission\ Zeros\ of\ G(s)\ = Roots\ of\ \prod_1^p \epsilon_i(s) \tag{7.4}$$

$$Poles\ of\ G(s)\ = Roots\ of\ \prod_1^p \psi_i(s) \tag{7.5}$$

Finding the zeros, through the Smith McMillan forms, can be summarized as follows

- Find the least common denominator of all of the denominators in $G(s)$. Call the resulting monic polynomial $d(s)$.
- Form $G(s) = \frac{1}{d(s)}P(s)$, where $P(s)$ is now a polynomial matrix.
- Through elementary operations - via unimodular matrices - transform $P(s)$ into the 'Smith' form (i.e., the Smith McMillan form with all of the denominators equal to 1).
- Divide diagonal entries by $d(s)$ to the Smith-McMillan form and read off the poles and zeros.
- For this process to be successful, the lowest power polynomial should be placed in the (1,1) element (by elementary operations).

Let us work an example in detail. We start with the following system

$$G(s) = \frac{1}{d(s)}P(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2s^2-8 \end{bmatrix}.$$

We start the elementary operations: First eliminate the (2,1) and (3,1) elements by adding a proper multiple of the first row to the second and third rows

$$P(s) = M_1^{-1}(s)M_1(s)P(s) = M_1^{-1}(s)P_1(s)$$

with

$$P_1(s) = \begin{bmatrix} 1 & -1 \\ 0 & 3(s^2-4) \\ 0 & 3(s^2-4) \end{bmatrix}.$$

Next, we eliminate the (3,2) element

$$P(s) = M_1^{-1}(s)M_2^{-1}(s)M_2(s)P_1(s) = M_1^{-1}(s)M_2^{-1}(s)P_2(s)$$

where

$$P_2(s) = \begin{bmatrix} 1 & -1 \\ 0 & 3(s^2-4) \\ 0 & 0 \end{bmatrix}.$$

Next, we eliminate the (1,2) element

$$P(s) = M_1^{-1}(s)M_2^{-1}(s)P_2(s)N_1(s)N_1^{-1}(s) = M_1^{-1}M_2^{-1}(s)P_3(s)N_1^{-1}(s)$$

where

$$P_3(s) = \begin{bmatrix} 1 & 0 \\ 0 & 3(s^2-4) \\ 0 & 0 \end{bmatrix}.$$

Lastly, we scale the second column through

$$\begin{aligned} P(s) &= M_1^{-1}(s)M_2^{-1}(s)P_3(s)N_2(s)N_2^{-1}(s)N_1^{-1}(s) \\ &= M_1^{-1}(s)\,M_2^{-1}(s)\,P_4(s)N_2^{-1}(s)\,N_1^{-1}(s) \end{aligned}$$

where

$$P_4(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s^2-4) \\ 0 & 0 \end{bmatrix}.$$

Note that

$$G(s) = \frac{1}{d(s)}P(s) = M_1^{-1}(s)M_2^{-1}(s)\frac{1}{d(s)}P_4(s)N_2^{-1}(s)N_1^{-1}(s)$$

or equivalently, from the definition, pre and post multiplying with $M_i(s)$ and $N_i(s)$,

$$G_1(s) = \frac{1}{d(s)}P_4(s), \quad M(s) = M_2(s)M_1(s), \quad N(s) = N_1(s)N_2(s).$$

As a result, we have poles at (-1,-1,-2) and zeros at (+2) (nonminimum phase zero).

7.3 Zeros and the State Space Approach

In this subsection, we will discuss finding transmission zeros from the state space representation of a system. Consider a system with

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{7.6}$$

Associated with this system, let us introduce the following 'system matrix'

$$S(s) = \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \tag{7.7}$$

Assumption 7.4. *The system is not* degenerate*, where by degenerate we mean the rank of S is strictly less than the minimum of $[n + rank(B)\ ,\ n + rank(C)]$ for all values of s. If the system is degenerate, one has to be careful. Most codes fail in such cases. This condition (i.e., nondegenerate) is often expressed as S having a normal rank equal to the minimum of $[n + rank(B)\ ,\ n + rank(C)]$.*

Definition 7.5. *Invariant zeros are those values of s that result in rank of S becoming less than the minimum of $[n + rank(B)\ ,\ n + rank(C)]$.*

Remark 7.6. *If there is direct feedthrough, i.e., $y = Cx + Du$, simply change the '0' in the lower right corner of $S(s)$ to D. The rest follows as before.*

Remark 7.7. *If the system is both controllable and observable (and non-degenerate), the invariant zeros are the same as the transmission zeros, discussed earlier. Indeed, there are only one kind of zeros. They can be calculated, for example if rank of B is smaller, by studying $Sx = 0$ and turning it into a generalized eigenvalue problem of the form $Mx = \lambda Nx$. Then standard eigenvalue problems can be used to solve for the eigenvalues (which are the invariant zeros).*

When the system is either uncontrollable or unobservable, the problem become more complicated. For example, let (A, C) have an unobservable mode; i.e., there exists a nonzero vector x such that $[Ax = \lambda x\ ,\ Cx = 0]$. Then this λ is known as an 'output decoupling' zero. 'Input' decoupling zeros are defined similarly. In such cases, system zeros are the transmission zeros plus the decoupling zeros.

Remark 7.8. *Decoupling zeros sometimes are invariant zeros, sometimes not!! Therefore, one has to be careful (see Homework Problems).*

Remark 7.9. *It is possible not to have any zeros (e.g., $\frac{1}{s^2}$).*

Remark 7.10. *Let there be m inputs, and l outputs. Then we have the following*

$$\begin{array}{l} number\ of\ zeros\ \leq n - \max(m,l)\ \ if\ \ D = 0 \\ number\ of\ zeros\ \leq n\ \ if\ \ D \neq 0 \\ number\ of\ zeros\ \leq n - m - d\ \ if\ \ D = 0\ \ m = l\ \ and \\ d = rank\, def\, ficiency of (CB) \end{array}$$

Remark 7.11. *Let the number of outputs be at least as large as the number of inputs (i.e., $l \geq m$), then*

$$\begin{array}{rcl} i)\ squaring\ down & : & s\ a\ zero\ of \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \\ & \Rightarrow & s\ a\ zero\ of \begin{bmatrix} sI - A & -B \\ FC & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{rcl} ii)\ squaring\ up & : & s\ a\ zero\ of \begin{bmatrix} sI - A & -BG \\ C & 0 \end{bmatrix} \\ & \Rightarrow & s\ a\ zero\ of \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{rcl} iii)\ s\quad a\quad zero\ of \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} & & \\ \Rightarrow & s\ a\ zero\ of \begin{bmatrix} sI - A - BKC & -B \\ KC & 0 \end{bmatrix} & \end{array}$$

7.4 PROBLEM SET

Exercise 7.12. *In the example used earlier, find the unimodular matrices, $M_1(s)$, $M_2(s)$, $N_1(s)$, $N_2(s)$, and their inverses. Finally, what are $M(s)$ and $N(s)$ that turn the transfer function into the Smith form?*

Exercise 7.13. *Using the Smith McMillan form, find the poles and zeros of the following transfer function $G(s)$*

$$\frac{1}{(s+1)(s+2)(s-1)} \begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix}$$

Exercise 7.14. *Consider the following system*

$$S(s) = \left[\begin{array}{ccc|c} s-1 & 0 & 0 & 0 \\ 0 & s+1 & 0 & 1 \\ 0 & 0 & s+3 & 1 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

Is this system controllable? Is it observable? Is there an input decoupling zero which is not an invariant zero?

Exercise 7.15. *Show the properties of the last remark are true. What are the corresponding results for the case of $m \geq l$?*

8 CONTROL SYSTEM DESIGN

8.1 Single Input Single Output Systems

For the SISO system shown in Figure 8.1, we use the standard notation:

- $r(s)$: The input command ($r(t)$)
- $K(s)$: The Compensator
- $G(s)$: The plant
- $u(s)$: The control or command ($u(t)$)
- $y(s)$: The output ($y(t)$)
- $d(s)$: The disturbance signal ($d(t)$)
- $n(s)$: The sensor noise ($n(t)$)
- $e(s)$: The error signal ($e(t)$)

Throughout this Section, we will the concern ourselves with the set up of Figure 8.1: i.e., the so called one degree of freedom framework. For the two degree of freedom approach (which yield similar but not identical results), consult the textbook by Wolovich. Also, the same text can be considered an excellent source for material covered in this Section. Simple manipulations yield the following

$$y(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} [r(s) - n(s)] + \frac{1}{1 + G(s)K(s)} d(s) \tag{8.1}$$

Definition 8.1. *For a variety of reasons, some of which will be discussed presently, we will use a few transfer functions repeatedly. These are:*

$$1 + G(s)K(s) = Return\ Difference\ Transfer\ Function \tag{8.2}$$

$$S(s) = \frac{1}{1 + G(s)K(s)} = Sensitivity\ Transfer\ Function \tag{8.3}$$

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} = Complimentary\ Sensitivity\ Trans.\ Func. \tag{8.4}$$

Note that the complimentary transfer function is simply the closed loop transfer function and $G(s)K(s)$ is just the feedforward loop gain. Also, note that $T(s) + S(s) = 1$.

For a good control system design, a variety of issues are taken into consideration. Let us consider a few of the more important ones

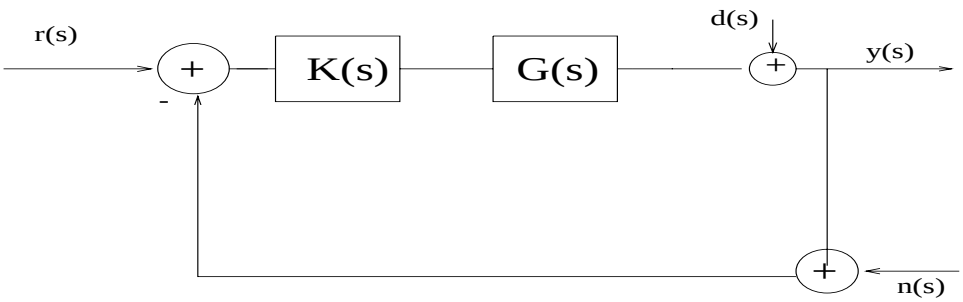

Figure 8.1: Block Diagram of a SISO System

1. Sensitivity to Modeling Error
2. Command Following
3. Disturbance Rejection
4. Noise Propagation
5. Stability Robustness

Clearly, this is not a complete list (leaves transient response out, for example), but it does cover most of the traditionally critical issues (or design specifications) faced in control system design. Also, the ordering is somewhat misleading. Naturally, stability is always the first priority of any control design.

I. Sensitivity to Modeling Error

Let us assume that the actual plant is of the form

$$G(s) = G^*(s) + \delta G(s)$$

where (*) does not mean adjoint! It simply means ideal, and $\delta G(s)$ reflects the error between the actual and nominal plants.

(i) open loop: The open loop output is

$$y(s) = G(s)\, r(s) = G^*(s)\, r(s) + \delta G(s) r(s) = y^*(s) + \delta y(s)$$

where $y^* = G^* r$ is output of the nominal plant and δy is the error between the nominal and actual outputs. Note that the normalized error has the following from

$$\frac{\delta y(s)}{\delta G(s)} = r(s) \quad or \quad \frac{\delta y(s)}{y^*(s)} = \frac{\delta G(s)}{G^*(s)} \tag{8.5}$$

(i) closed loop: For the closed loop, we have the following . Note that the dependence on 's' is dropped for brevity

$$y = \frac{GK}{1+GK}\, r = \frac{(G^* + \delta G)K}{1 + (G^* + \delta G)K}\, r = y^* + \delta y$$

where y^* is the closed loop response of the ideal -nominal - system; i.e.,

$$y^* = \frac{G^* K}{1 + G^* K}\, r$$

and $\delta y = y - y^*$. Next,

$$1 + \frac{\delta y}{y^*} = \frac{y}{y^*} = \frac{G(1 + G^* K)}{(1 + GK)G^*}.$$

As a result, after simplification, we have

$$\frac{\delta y}{y^*} = \frac{\delta G}{G^*} . \frac{1}{1 + GK} \tag{8.6}$$

Remark 8.2. *The error in the output, due to error in modeling, is modified by the sensitivity transfer function (compare* (8.5) *with* (8.6)*). As a result, for the frequencies where δG is 'large', relatively speaking, a small sensitivity (or a 'large' GK) can be used to reduce the closed loop sensitivity (compared to the open loop sensitivity). Notice that we are not discussing stability - only sensitivity.*

II. Command Following

To study command following, set the other inputs to zero; i.e., $n = d = 0$. We then have

$$y = \frac{GK}{1 + GK}\, r$$

If $(1 + GK)$ is 'large', so is - or must be - GK. Therefore, their ratio is approximately one.

Remark 8.3. *Over the frequencies where following r is desired, we want y to be close to r, or the closed loop transfer function be close to one. As a result, 'large' GK is needed. Often, the command signals are in the low frequencies.*

III. Disturbance Rejection

Let $n = r = 0$, to focus of effects of d on the output. We then have

$$y = \frac{d}{1 + GK} \quad or \quad e = \frac{-1}{1 + GK}\, d$$

Remark 8.4. *Over the frequencies where there are large disturbances, it is desirable to have GK large, to reduce their effects . In most cases, such disturbances are in the low frequencies .*

IV. Effects of Noise

We set $r = d = 0$, to focus on noise (n) in the output measurements. It is easy to see that

$$y = -\frac{GK}{1+GK}\, n$$

In this case, note that we want the effects of noise on the output to be small. Note that this requires that GK be small.

Remark 8.5. *Over the frequencies where there is significant sensor noise (typically high frequencies), we need 'small' GK. As a result, 'small' GK is needed.*

V. Stability and Robustness

Naturally, we require nominal stability (i.e., the nominal closed loop is stable). This is the minimum requirement, of course. The more interesting issue is the stability of closed loop system is the actual model is not the same as the nominal one.

Let the plant be in the following form

$$G(s) = G^*(s)\ [1 + L(s)]$$

which is called the form with the multiplicative uncertainty . Note that one can always use $L(s) = \frac{\delta G(s)}{G^*(s)}$. We want to study the stability of the closed loop system of the actual plant, for a range of $L(s)$. Since we will assume very little knowledge of $L(s)$, we will rely on the Nyquist plots. To this end, we will assume the following

Assumption 8.6. *The number of unstable poles of $G(s)$ - the actual plant - is the same as the number of unstable poles of the nominal plant $G^*(s)$.*

Assumption 8.7. *An upper bound for the magnitude of the uncertainty is available; i.e, a function $l(w)$ is available such that $|L(jw)| \leq l(w)$. No other assumptions is made regarding $L(s)$.*

Assumption 8.8. *The controller, $K(s)$ is chosen so that the nominal closed loop system is stable.*

Assumption 8.6 is needed so that the encirclement count of the Nyquist plots remains constant over all allowable uncertainty. Recall the following from the standard SISO systems.

Definition 8.9. *The closed loop system is stable if $K(jw)G(jw)+1 \neq 0$ and the encirclements (of plots of $K(jw)G(jw)$) around $(-1, 0)$ point (on the complex plain) is not changed -for all allowable $L(s)$- from that of the nominal system (i.e., $K(jw)G^*(jw)$).*

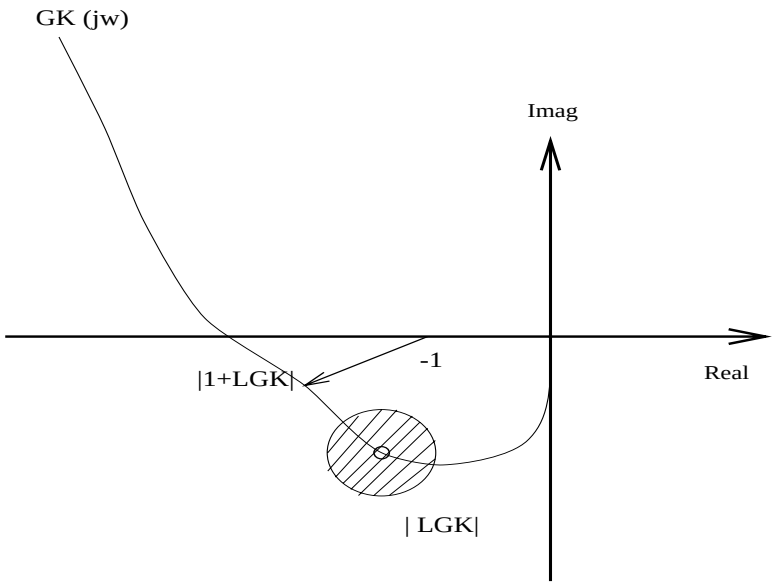

Figure 8.2: Nyquist plot for multiplication uncertainty

One way, to guarantee that a change in the encirclement count is avoided, is by noting that we need

$$1 + K(jw)G^*(jw) + K(jw)G^*(jw)L(jw) \neq 0$$

and thus by requiring the following

$$|L(jw)G^*(jw)K(jw)| \leq |1 + G^*(jw)K(jw)| \tag{8.7}$$

which is satisfied if the following holds

$$|L(jw)| \leq l(w) \leq \frac{|1 + G^*(jw)K(jw)|}{|G^*(jw)K(jw)|} \tag{8.8}$$

or

$$\frac{|K(jw)G^*(jw)|}{|1 + K(jw)G^*(jw)|} < \frac{1}{l(w)}. \tag{8.9}$$

Remark 8.10. *In practice, good models can be obtained over low frequencies. As a result, typical form of $l(w)$ is a curve that grows and becomes large as w increases. On large frequencies, therefore, the nominal transfer function must be very small, as a result, small KG^* are desired over high frequencies (or frequencies where large modeling errors exist).*

As a result of the above discussion, we arrive at the typical shape that we expect the Bode plot to have (see Figure 8.3). The exact cut-offs and lower and higher bound, of course, depend on the exact specifications of the control system.

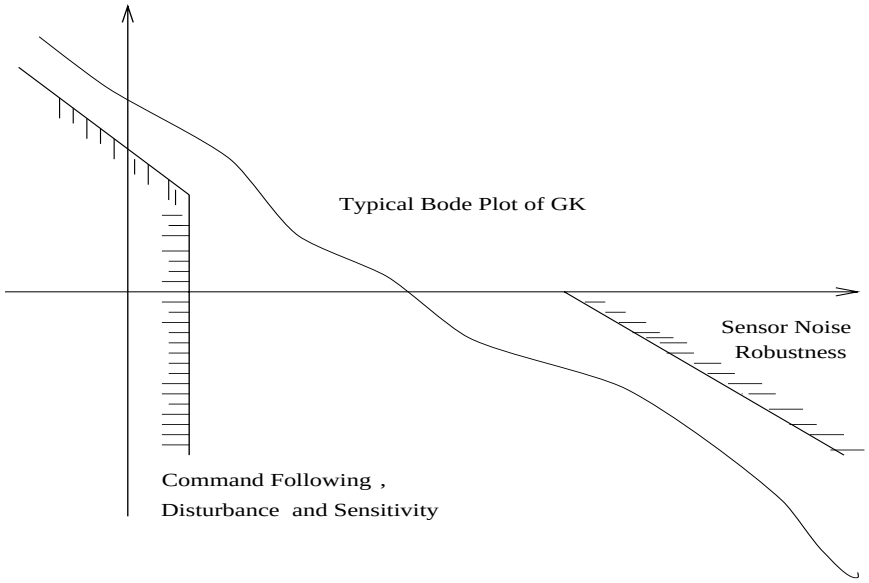

Figure 8.3: A typical desirable Bode Plot

8.2 Multi Input Multi Output Systems

We go parallel to the SISO case (as far as we can). The block diagram will look the same, for example. We will use upper case letters for signals $U(s)$, $Y(s)$, etc. to underline the fact that they are vectors. We start by using

$$E(s) = R(s) - Y(s) - N(s) \;\; and \;\; Y(s) = D(s) + G(s)K(s)E(s)$$

Remark 8.11. *In MIMO systems, $K(s)G(s) \neq G(s)K(s)$, and you need to be careful. Typically, they appear in reverse order of the block diagram; i.e.*

$$U(s) = K(s)E(s) \;\; and \;\; Y(s) = G(s)U(s) + D(s) = G(s)K(s)E(s) + D(s)$$

Next, we have

$$Y(s) = D(s) + G(s)K(s)R(s) - G(s)K(s)Y(s) - G(s)K(s)N(s)$$

or solving for $Y(s)$

$$Y(s) = [I+G(s)K(s)]^{-1}G(s)K(s)[R(s)-N(s)]+[I+G(s)K(s)]^{-1}D(s) \quad (8.10)$$

where similar to the SISO systems we can define

$$I + G(s)K(s) = Return\ Difference\ Transfer\ Function\ (at\ the\ output) \quad (8.11)$$

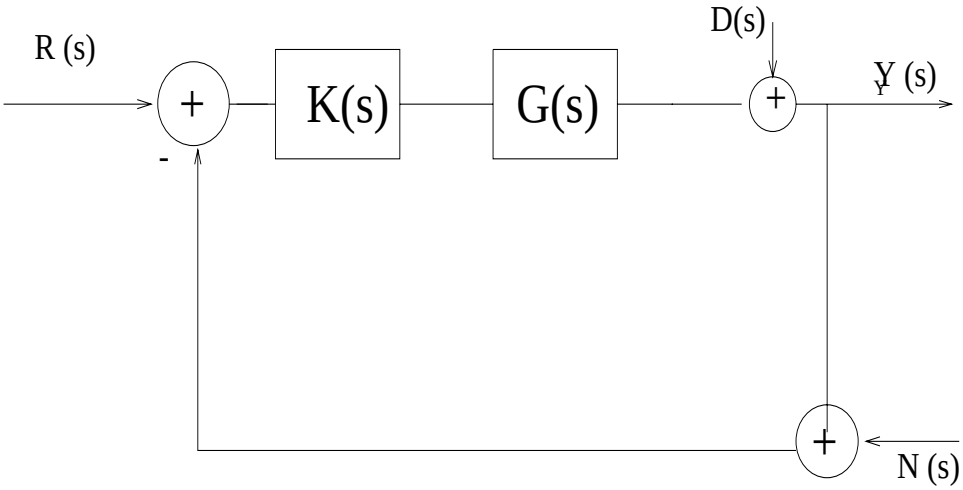

Figure 8.4: Block Diagram of a MIMO System

$$S(s) = [I + G(s)K(s)]^{-1} = Sensitivity\ Transfer\ Function\ (at\ the\ output) \tag{8.12}$$

$$\begin{aligned} T(s) &= [I + G(s)K(s)]^{-1}G(s)K(s) \\ &= Complimentary\ Sensitivity\ Transfer\ Function \end{aligned} \tag{8.13}$$

As before, it is easy to see that $T(s)$ is the closed loop transfer function and

$$T(s) = S(s)\,G(s)\,K(s), \qquad T(s) + S(s) = I \tag{8.14}$$

Similar to the SISO case, we will study the following issues

1. Sensitivity to Modeling Error
2. Command Following
3. Disturbance Rejection
4. Noise Propagation
5. Control Effort
6. Stability Robustness

Before going through each case, let us discuss the notion of 'large' and 'small' for MIMO systems. In the SISO case, everything was relatively simple; we took the absolute value of the transfer function, evaluated on the imaginary axis.

That is, we looked at $|G(jw)|$, which is consistent with standard notion of Bode plots (input at that frequency gets magnified by the size of the transfer function).

For MIMO system, we also consider what the transfer function does to the incoming signal. We call a transfer function 'large' if it magnifies the amplitude of signal, and 'small' if it does the reverse. The problem is that our inputs and outputs are vectors themselves.

From now on, a signal is considered large if its 2-norm (in frequency or time domain - recall Parseval's Theorem) is large. Therefore, we are interested in seeing what does a given transfer function do to the 2-norm of its input. Now considering the basic properties of the singular value, we have

$$\begin{aligned} Y(jw) &= G(jw)U(jw) \Rightarrow \underline{\sigma}[G(jw)]\ \|U(jw)\|_2 \leq \|Y(jw)\|_2 \\ &\leq \overline{\sigma}[G(jw)]\ \|U(jw)\|_2 \end{aligned}$$

where the maximum (or minimum) is over all frequencies. A 'small' transfer function, therefore, has a small maximum singular value and a 'large' transfer function has a large minimum singular value. These singular values indicate the best and worst case magnification a signal (at a given frequency) may experience. Also, due to Parseval's Theorem, these principle gains can be interpreted at L_2 gains for the convolutions (e.g., given a signal with unit energy what are the upper and lower bounds on the energy of the output).

With this in mind, we can go over the design issues listed above.

I. Sensitivity to Modeling Error

Let us assume that the actual plant is of the form

$$G(s) = G^*(s) + \Delta G(s)$$

where $G^*(s)$ is the nominal or ideal plant , and $\Delta G(s)$ reflects the error between the actual and nominal plants. The following can be shown relatively easily (see the Problem Set!)

(i) open loop

$$\Delta Y(s) = \Delta G(s)\ [G^*(s)]^{-1}\ Y^*(s) \tag{8.15}$$

(i) closed loop

$$\Delta Y(s) = [I + G(s)K(s)]^{-1}\ \Delta G(s)\ [G^*(s)]^{-1}\ Y^*(s) \tag{8.16}$$

Remark 8.12. *The error in the output, due to error in modeling, is modified by the sensitivity transfer function (compare* (8.15) *with* (8.16)*. As a result, for the frequencies where ΔG is 'large', we have large open loop sensitivity. To reduce this sensitivity, the sensitivity transfer function can be exploited.*

To reduce sensitivity, we require the maximum singular value of $S(s)$ be small. Recalling the property that $\overline{\sigma}(A) = \frac{1}{\underline{\sigma}(A^{-1})}$, we arrive at the requirement that minimum singular value of the return difference transfer function be large. Recalling the property that $\underline{\sigma}(I + A)$ is between $(\underline{\sigma}(A) - 1)$ and $(\underline{\sigma}(A) + 1)$, we conclude that $\underline{\sigma}(G(s)K(s))$ should be large, over frequencies of interest (i.e., where we want little sensitivity). Often, such a requirement is written as requiring

$$\underline{\sigma}[G(jw)K(jw)] \geq p(w). \tag{8.17}$$

The function $p(w)$ can thus be used to set the levels and ranges of the frequencies of interests (e.g., $p(w)$ large at low frequencies, to ensure little sensitivity to modeling errors in such a range).

II. Command Following

As before, to study command following, set the other inputs to zero; i.e., $N = D = 0$. We then have

$$Y(s) = T(s)\ R(s)\ .$$

Therefore for good command following over the frequencies of interest (i.e., most often low frequencies), we need $T(s)$ be approximately identity. Since $I = T(s) + S(s)$, this requires very small $S(s)$. The rest follows as before and we end up with requirement of high minimum singular value for $G(jw)K(jw)$.

Remark 8.13. *Over the frequencies where following R is desired (i.e, typically low frequencies), large $\underline{\sigma}[GK]$ is desired.*

III. Disturbance Rejection

Let $N = R = 0$, to focus of effects of D on the output. Considering (8.10), to reduce the effects of disturbance on the output, $S(s)$ should be small, over the range of disturbance frequencies. Similar to the sensitivity analysis, we will need large $\underline{\sigma}[G(jw)K(jw)]$, for the range of frequencies with large disturbance (typically, low frequencies).

Next, we consider cases that may require small $\overline{\sigma}[G(jw)K(jw)]$.

IV. Effects of Noise

We set $R = D = 0$, to focus on noise (N) in the output measurements. Again, we examine (8.10). We have seen that large $\underline{\sigma}[GK]$ results in $T(s) \approx I$, which is certainly undesirable for those frequencies with large noise component. To reduce noise, we need to make $T(s)$ as small as possible (over these noisy frequencies). That is, we want $\overline{\sigma}[T(jw)]$ be very small. Since $\overline{\sigma}(AB) \leq \overline{\sigma}(A)\overline{\sigma}(B)$, we have

$$\overline{\sigma}[T(jw)] \leq \overline{\sigma}[S(jw)]\overline{\sigma}[G(jw)K(jw)] = \frac{\overline{\sigma}[G(jw)K(jw)]}{\underline{\sigma}[I + G(jw)K(jw)]} \tag{8.18}$$

Since $\underline{\sigma}(I + A)$ is between $(\underline{\sigma}(A) - 1)$ and $(\underline{\sigma}(A) + 1)$, the fraction on the right hand side of (8.18) can be made small if we make $\overline{\sigma}[G(jw)K(jw)]$ very small (see the Problem Set).

Remark 8.14. *Over the frequencies where there is significant sensor noise (typically high frequencies), we need 'small' GK. As a result, small $\overline{\sigma}[GK]$ is needed.*

V. Control Energy

One can show that

$$U(s) = K(s)S(s)[R(s) - N(s) - D(s)] \tag{8.19}$$

So to reduce control energy, one would seek small $\overline{\sigma}[K(s)S(s)]$. Dropping the dependency of 's' for brevity, we can rewrite this as

$$\overline{\sigma}[K[I + GK]^{-1}] \leq \overline{\sigma}(K)\overline{\sigma}[I + GK]^{-1} \leq \frac{\overline{\sigma}(K)}{\underline{\sigma}[I + GK]}.$$

The goal now is to reduce the RHS of the above equation, which implies needing

$$\underline{\sigma}[I + GK] \gg \overline{\sigma}(K).$$

Now, since $\underline{\sigma}[I + GK] \leq \overline{\sigma}[I + GK] \leq I + \overline{\sigma}(K)\overline{\sigma}(G)$, we must be seeking

$$I + \overline{\sigma}(K)\overline{\sigma}(G) \gg \overline{\sigma}(K) \quad or \quad \frac{1}{\overline{\sigma}(K)} + \overline{\sigma}(G) \gg 1. \tag{8.20}$$

Now consider two regions and the typical behavior of $G(jw)$ in these regions:
a) Low Frequencies: $G(jw)$ is large, which implies (8.20) can be realized.
b) High Frequencies: $G(jw)$ is small, which implies (8.20) can be realized if $\overline{\sigma}(K)$ is very small. Since $\overline{\sigma}(GK) \leq \overline{\sigma}(G)\overline{\sigma}(K)$, the need for low control energy requires a low $\overline{\sigma}(GK)$ over high frequencies.

In summary, for small control effort, the maximum singular value of GK should be small at the high frequencies. Naturally, similar results could have been obtained in the SISO case, as well.

Remark 8.15. *A similar result can be obtained by the following: Assume square and invertible G and K, and show that high $\underline{\sigma}(GK)$ results in an approximate relationship $U \approx G^{-1}[R - N - D]$. Therefore, any reference, disturbance or noise at high frequency results in very high control energy.*

VI. Stability Robustness

As you might expect, we will be following the development of the SISO systems. First, however, we need to review the MIMO version of the Nyquist criterion

Definition 8.16. *$G(s)$ is stable if the number of the encirclements (around the origin) of the map $det[I + G(jw)K(jw)]$, evaluated on the Nyquist D-contour, is equal to the negative of the number of unstable open-loop modes (poles) of $G(jw)$.*

We start by assuming the following relationship between the actual plant (i.e., $G(s)$), and nominal plant (i.e., $G^*(s)$).

$$G(s) = [I + L(s)]\ G^*(s)\ . \tag{8.21}$$

We are modeling the uncertainty as multiplicative *at the output*. Note that for MIMO systems, at the output and at the input are not the same. As with the SISO system, we make the following assumptions:

Assumption 8.17. *The nominal closed loop system is stable (meets the MIMO Nyquist criterion and determinant of $I + G^*K]$ is not zero at any frequency) and the number of unstable poles $G(s)$ - the actual plant - is the same as the number of unstable poles of the nominal plant $G^*(s)$.*

Assumption 8.18. *An upper bound for the magnitude of the uncertainty is available; i.e, a function $l(w)$ is available such that*

$$\overline{\sigma}[L(jw)] \leq l(w) \qquad \forall w.$$

.

As before, we will simplify notation by dropping 'jw'. For stability, the assumptions above imply that it is sufficient to have $det[I + (I + L)G^*K] \neq 0$ for all allowable $L(jw)$. Therefore, we need

$$\underline{\sigma}[I + (I + L)G^*K] > 0 \qquad \forall w\ . \tag{8.22}$$

Note, however, that

$$I + (I + L)G^*K = I + G^*K + LG^*K = [I + LG^*K(I + G^*K)^{-1}](I + G^*K)$$

assuming the inverse exist (Not a big assumption: nominal stability). Next,

$$\underline{\sigma}[I + (I + L)G^*K] \geq \underline{\sigma}(I + G^*K)\ \underline{\sigma}[I + LG^*K(I + G^*K)^{-1}]$$

where, by nominal stability, the first term on the right is nonzero. As a result, we focus on the second term and see if we can make it nonzero. Clearly (see Problem Set), this term is nonzero if we require

$$\underline{\sigma}(I) = 1 > \overline{\sigma}[LG^*K(I + G^*K)^{-1}]$$

or equivalently if

$$1 > \overline{\sigma}(L)\overline{\sigma}[G^*K(I + G^*K)^{-1}] \geq \overline{\sigma}[LG^*K(I + G^*K)^{-1}]$$

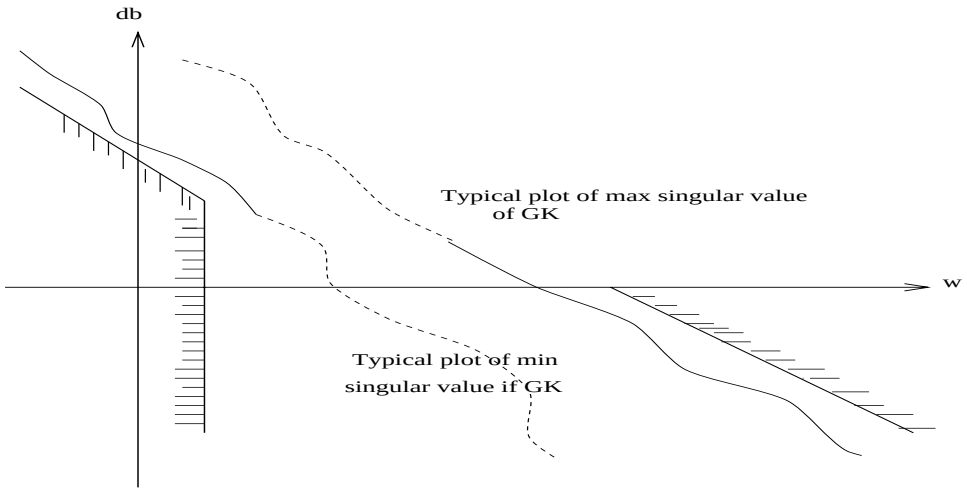

Figure 8.5: Max and Min singular value plots

Since $l(w) \leq \overline{\sigma}(L)$, then (8.22) holds if

$$1 > l(w)\overline{\sigma}[G^*K(I+G^*K)^{-1}] \geq \overline{\sigma}[LG^*K(I+G^*K)^{-1}]$$

or

$$\frac{1}{l(w)} > \overline{\sigma}[G^*K(I+G^*K)^{-1}]. \tag{8.23}$$

So far so good. Since in high frequencies, we have $l(w) \gg 1$, stability robustness - i.e., (8.23) - dictates a small maximum singular value for the nominal transfer function (note the very small trick involved). From previous development, we know that this requirement is the same as having small $\overline{\sigma}(GK)$ over high frequencies.

As a result of the above development, we arrive at the plots in Figure 8.5 for the max and min singular values of GK. Finally, before we leave this Section, we can study a few interesting issues:

1. Invert (8.23) to get

$$l(w) < \frac{1}{\overline{\sigma}[G^*K(I+G^*K]^{-1}]} = \underline{\sigma}[I+(G^*K)^{-1}]. \quad (why?) \tag{8.24}$$

The right hand side, therefore, defines the amount of modeling error a given nominal closed loop system can tolerate. In can be interpreted as the MIMO version of gain margin!!

2. What about performance in presence of uncertainty?

Earlier, we discussed sensitivity and model following, and other low frequency requirements in terms of $G(s)$, the actual or nominal. In many applications this may be acceptable since most modeling errors occur in high frequencies. However, if there were low frequency model error; i.e., $l(w)$ was not zero at low frequencies, we need to review the results.

Equation (8.17) implies that

$$\underline{\sigma}[I + G(jw)K(jw)] \geq p(w) \qquad G(jw) = (I + L(jw))G^*(jw)$$

while by (8.23)

$$\overline{\sigma}[G^*K(I + G^*K)^{-1}] < \frac{1}{l(w)}.$$

Now assume that $l(w) \leq 1$ in the low frequencies (i.e., nonzero, but moderate levels of uncertainty at low frequencies). Then

$$p(w) \leq \underline{\sigma}[I + (I + L)G^*K] = \underline{\sigma}\{[I + LG^*K(I + G^*K)^{-1}][I + G^*K]\} \quad (8.25)$$

using $\underline{\sigma}(AB) \geq \underline{\sigma}(A)\underline{\sigma}(B)$, (8.25) is satisfied if

$$p(w) \leq \underline{\sigma}[I + LG^*K(I + G^*K)^{-1}]\underline{\sigma}[I + G^*K]. \quad (8.26)$$

Now, if we assume that the nominal system has high gains at low frequencies, we can use $\underline{\sigma}[G^*K] \gg 1$ in low frequencies, which implies $\underline{\sigma}[I+G^*K] \approx \underline{\sigma}[G^*K]$, with G^*K nonsingular and $T(jw) \approx I$. Equation (8.26) is therefore equivalent to

$$p(w) \leq \underline{\sigma}(I + L)\ \underline{\sigma}(G^*K). \quad (8.27)$$

Since $\underline{\sigma}(AB) \geq \underline{\sigma}(A) - \overline{\sigma}(B)$, (8.27) must be satisfied if

$$p(w) \leq (1 - l(w))\ \underline{\sigma}[G^*K]$$

or

$$\frac{p(w)}{1 - l(w)} \leq \underline{\sigma}[G^*K] \quad (8.28)$$

that is, the modeling uncertainty at low frequencies makes the job of the controller (for performance type measures) more difficult.

8.3 A Few Matrix Identities

Let A and C be nonsingular matrices of possibly different dimensions. If there are, possibly nonsquare, matrices B and D such that BCD is the same dimension as A, then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}. \quad (8.29)$$

The matrix inversion lemma can be applied to a variety of cases. Some, exploit the fact that the dimension of the matrices that require inversion may be smaller on the right hand side. Consider the following - where matrices G_1, G_2, etc. can be either constant of functions of time or 's'

$$\begin{aligned}[I_n + G_2\,G_1\,H_2\,H_1]^{-1}G_2\,G_1 &= G_2[I_m + G_1H_2H_1G_2]^{-1}G_1 \quad (8.30)\\ &= G_2G_1[I_r + H_2H_1G_2G_1]^{-1} \quad (8.31)\\ &= G_2G_1 - G_2G_1H_2[I_p + H_1G_2G_1H_2]^{-1}H_1G_2G_1\end{aligned}$$

$$(P^1 + KC)^{-1} = P - PK(I + CPK)^{-1}CP \quad (8.32)$$

$$(I + KCP)^{-1} = I - K(I + CPK)^{-1}CP \quad (8.33)$$

$$(I + PKC)^{-1} = I - PK(I + CPK)^{-1}C \quad (8.34)$$

$$(I + G)^{-1} + (I + G^{-1})^{-1} = I \quad for\ G\ square\ and\ invertible \quad (8.35)$$

$$\overline{\sigma}[(I + G)^{-1}] + \overline{\sigma}[(I + G^{-1})^{-1}] \geq 1 \quad (8.36)$$

$$\overline{\sigma}[(I + G)^{-1}] + 1 \geq \overline{\sigma}[(I + G^{-1})^{-1}] \quad (8.37)$$

$$\overline{\sigma}[(I + G^{-1})^{-1}] + 1 \geq \overline{\sigma}[(I + G)^{-1}] \quad (8.38)$$

$$\overline{\sigma}(G) \geq \frac{\underline{\sigma}(I + G)}{\underline{\sigma}(I + G^{-1})} \geq \underline{\sigma}(G) \qquad (Use\ \ \underline{\sigma}(AB) \leq \underline{\sigma}(A)\overline{\sigma}(B)\ \ ) \quad (8.39)$$

8.4 PROBLEM SET

Exercise 8.19. *Show (or prove of whatever) properties P1-P3 of singular values.*

Exercise 8.20. *Show (or prove of whatever) properties P4-P13 of singular values.*

Exercise 8.21. *Show or derive equations* (8.15) *and* (8.16).

Exercise 8.22. *Show* $(I + GK)^{-1}GK = I - (I + GK)^{-1}$.

Exercise 8.23. *Show that* $\overline{\sigma}[G(jw)K(jw)] \ll 1$ *implies* $\overline{\sigma}[T(jw)] \ll 1$, *where* $T(s)$ *is the closed loop transfer function.*

Exercise 8.24. *Derive equation* (8.19).

Exercise 8.25. *Let* $A > 0$. *Show that if* $\underline{\sigma}(A) > \overline{\sigma}(B)$, *then* $A + B > 0$.

Exercise 8.26. *Do as many as the matrix identities of the previous section.*

9 LOOP SHAPING AND RECOVERY

In this Section, we will look for ways to obtain desirable loop shapes (as discussed in the previous Section) by the LQR technique. We will also look into effects of observers of the closed loop shapes and ways to 'recover' the desirable loop shapes we can get with LQR, when observers are needed (i.e., we cannot measure the full state).

9.1 The KALMAN Equality

Let us start with the steady state Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{9.1}$$

and the optimal gain matrix

$$u = -Kx, \quad K = R^{-1}B^T P. \tag{9.2}$$

From (9.2), it is clear that the following identities hold

$$R^{\frac{1}{2}}K = R^{-\frac{1}{2}}B^T P \tag{9.3}$$

$$K^T RK = PBR^{-1}B^T P. \tag{9.4}$$

The Riccati equation of (9.1) can be written as

$$PA + A^T P - K^T RK + Q = 0,$$

or after some manipulations

$$P(sI - A) + (-sI - A^T)P + K^T RK = Q. \tag{9.5}$$

Now after multiplying (9.5), on the left, by $R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}$ and, on the right, by $(sI - A)^{-1}BR^{-\frac{1}{2}}$, we will get

$$\begin{aligned} &R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}P(sI - A)(sI - A)^{-1}BR^{-\frac{1}{2}} \\ &+R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}(-sI - A^T)P(sI - A)^{-1}BR^{-\frac{1}{2}} \\ &+R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}K^T RK(sI - A)^{-1}BR^{-\frac{1}{2}} \\ &= R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}Q(sI - A)^{-1}BR^{-\frac{1}{2}} \ , \end{aligned}$$

or, considering (9.3) and (9.4),

$$\begin{aligned} &R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}K^T R^{\frac{1}{2}} + R^{\frac{1}{2}}K(sI - A)^{-1}BR^{-\frac{1}{2}} \\ &+R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}K^T RK(sI - A)^{-1}BR^{-\frac{1}{2}} \\ &= R^{-\frac{1}{2}}B^T(-sI - A^T)^{-1}Q(sI - A)^{-1}BR^{-\frac{1}{2}} \ . \end{aligned}$$

The left hand side of this equation is of the form $Y^T + X + Y^T X$. Adding I to both side and noting that $(I+Y)^T(I+X) = I + Y^T + X + Y^T X$, we have

$$\begin{aligned}[I + R^{\frac{1}{2}} K(-sI - A)^{-1} B R^{-\frac{1}{2}}]^T [I + R^{\frac{1}{2}} K(sI - A)^{-1} B R^{-\frac{1}{2}}] = \\ I + R^{-\frac{1}{2}} B^T (-sI - A^T)^{-1} Q (sI - A)^{-1} B R^{-\frac{1}{2}}.\end{aligned} \tag{9.6}$$

Equation (9.6) is known as the Kalman equality (the better known Kalman inequality is very similar; i.e. Left Hand Side (LHS) of (9.6) $\geq I$, for all w, where $s = jw$!).

9.2 Asymptotic behavior of Riccati equation

The book by Kwakernaak and Sivan is the primary source of this section (as well as a paper by B. Francis). Again consider the Riccati equation of (9.1), except we let

$$R = \rho^2 N, \quad \textit{and} \;\; Q = H^T H \tag{9.7}$$

without any loss of generality. We will study the behavior of the positive definite solution of (9.1) as ρ tends to infinity and zero.

Result A: $\rho \to 0$
The limit:

$$\lim_{\rho \to 0} P_\rho = P_o$$

exists. Also, let z_i's, $i = 1, 2, \ldots p$, be the transmission zeros of the system $H(sI - A)^{-1}B$ (which is assumed to be controllable and observable). Then the closed loop poles approach z_i's if z_i is on the closed left half plane (and to $-z_i$ if is on the right half plane). The remaining $(n - p)$ poles go to infinity in a Butterworth pattern.

Lastly, if the transfer function $H(sI - A)^{-1}B$ **is right invertible with no unstable zeros**, then $P_o = 0$. Furthermore, is this case, for some unitary W (i.e, $WW^T = I$)

$$K = R^{-1} B^T P \to R^{-\frac{1}{2}} W H = \frac{1}{\rho} N^{-\frac{1}{2}} W H \tag{9.8}$$

Result B: $\rho \to \infty$
In this case the closed loop pole approach the open loop poles (if stable) or their mirror image with respect to the imaginary axis (if unstable). If the open loop poles are stable, the gain matrix approaches zero.

9.3 Good LQR loop shapes

Recall that for robustness and performance, we would like to shape KG according to some specifications. For MIMO systems, this 'shaping' corresponds to limiting, from below, the minimum singular value of KG or GK over low frequencies and from above, the maximum singular value of KG, over high frequencies. In the following, we will review a method that uses many of the topics we have covered in class.

For simplicity, we make the following assumptions

Assumption 9.1. *Without any loss of generality, we assume that*

$$R = \rho^2 I, \qquad Q = H^T H. \tag{9.9}$$

Recall that for the LQR problem

$$K(s)G(s) = K(sI - A)^{-1}B = K\Phi(s)B \tag{9.10}$$

where

$$K = R^{-1}B^T P = \frac{1}{\rho^2}B^T P, \quad \textit{and} \quad \Phi(s) = (sI - A)^{-1}. \tag{9.11}$$

Since we are concerned with frequency characteristics of the system, in the remainder if this section we will replace s by jw.

For the particular choice of Q and R used here, the Kalman equality of (9.6) becomes

$$\begin{aligned} [I \quad &+ \quad K(jwI - A)^{-1}B]^h [I + K(jwI - A)^{-1}B] \\ &= \quad I + \frac{1}{\rho^2}[H(jwI - A)^{-1}B]^h [H(jwI - A)^{-1}B] \end{aligned} \tag{9.12}$$

where *the superscript 'h' denotes complex conjugate transpose.* Using (9.11), we can rewrite (9.12) as

$$[I + KG]^h [I + KG] = I + \frac{1}{\rho^2}[H\Phi B]^h [H\Phi B] \tag{9.13}$$

which implies

$$\lambda_i \left([I + KG]^h [I + KG]\right) = 1 + \frac{1}{\rho^2} \lambda_i \left([H\Phi B]^h [H\Phi B)\right.$$

which, considering the basic definition of singular values, reduces to

$$\sigma_i([I + KG]) = \sqrt{1 + \frac{1}{\rho^2} \sigma_i^2 ([H\Phi B])}. \tag{9.14}$$

Equation (9.14) can be used for low frequency performance. Recall that we typically need very large minimum singular values for KG over these frequencies. Therefore a good approximation from, (9.14), over these low frequencies, is

$$\sigma_{min}(KG) \approx \frac{1}{\rho}\sigma_{min}(H\Phi B). \tag{9.15}$$

As a result, using the free design parameters, i.e., H and ρ, one can increase the minimum singular value. Note that this method is approximate; i.e., equation (9.15) is used to get initial estimates for H and ρ, that satisfy the performance requirements. These estimates are used in the Riccati equation to obtain exact values. (Also, note the inherent trial and error nature). Finally, the exact form of H can be used to make the minimum and maximum singular values closer to one another (particularly over the cross over region).

For robustness, we need to investigate the following conditions

$$\frac{1}{\ell_m(w)} > \overline{\sigma}(I + KG)^{-1}KG, \quad \forall\ w \tag{9.16}$$

or, equivalently, (recall the last Section)

$$\underline{\sigma}(I + [KG]^{-1}) > \ell_m(w). \tag{9.17}$$

From (9.14), it is clear that

$$\underline{\sigma}(I + [KG]) > 1, \quad \forall\ w. \tag{9.18}$$

A few decades ago, Alan Laub showed that (9.18) implies that

$$\underline{\sigma}(I + [KG]^{-1}) > \frac{1}{2},\ \forall\ w \tag{9.19}$$

that is, the LQR controller provides a small robustness bounds for all frequencies automatically. Typically, the bound on the uncertainty ($\ell_m(w)$) becomes larger at high frequencies (considerably larger than 1). In high frequency range, therefore, we need to be concerned with the behavior of the maximum singular value of KG (recall the previous Section).

Now, we will need the following assumption

Assumption 9.2. *The transfer function $H(sI-A)^{-1}B$ is minimum-phase and right invertible.*

From the asymptotic behavior of the Riccati equation, we have (recall that we are using $N = I$ in (9.8))

$$K_c \to \frac{1}{\rho}WH, \quad as \quad \rho \to 0 \tag{9.20}$$

where W is an orthogonal matrix. Finally

$$K_cG = K_c(sI - A)^{-1}B = K_c\frac{1}{s}(I - \frac{A}{s})^{-1}B$$

At high frequencies, (i.e., $s = jw$ and w very large), as ρ approaches zero, we can approximate KG by

$$K_cG \longrightarrow \approx \frac{WHB}{jw\rho}, \quad i.e., \quad \overline{\sigma}(K_cG) \longrightarrow \approx \overline{\sigma}(\frac{HB}{jw\rho}) \tag{9.21}$$

Equation (9.21) can be used for robustness synthesis. Using (9.21) as an approximate value for high frequencies (and small values of ρ), the parameters H and ρ can be sought that satisfy both the performance requirements and the robustness requirements; i.e., keeping the maximum singular value of KG small at high frequencies and minimum singular value of KG large at low frequencies!

While a very small ρ would guarantee the satisfaction of (9.15), it might result in violation of the high frequency requirement (via (9.21)), over some of the intermediate frequencies. A typical consideration is the crossover frequency. The crossover frequencies are defined at those frequencies that result in $\sigma_i(KG) = 1$, for some i. Let w_{cmax} correspond to the maximum singular value of KG becoming one (note that $w_{cmax} \approx \frac{\overline{\sigma}(HB)}{\rho}$). After some thought (and maybe a touch of voo-doo), it becomes clear that this w_{cmax} should be not much larger (or preferably even smaller) than w_l (i.e., the frequency where ℓ_m becomes 1).

Another interesting point is that (9.21) implies that KG rolls off at, approximately, 20 db rate. If $\ell_m(w)$ grows faster than that, the crossover frequencies should be moved to the left.

9.4 Loop Transfer Recovery - LTR

So far, we have seen that LQR can be manipulated so that a desirable loop is obtained. Actually, other interesting results have been proven. For example, the concept of gain and phase margins have been extended to the MIMO case and LQR is shown to have infinite gain margin and sixty degrees phase margin. Due to our time constraint, we will omit these.

In 1978, John Doyle showed that all these nice margins can be lost once the observer is added (i.e., LQG does not necessarily have the same margins as the LQR). Around 1981, Doyle, along with Gunter Stein, followed this line by showing that the loop shape (of GK or KG) itself will, in general, change when the filter is added for estimation . well? What to do? where to go? The basic idea is very simple: suppose the LQR design has resulted in a great loop shape. What can be done (and indeed if) to recover the nice loop shape, if an observer is needed.

9.4.1 The Basic Idea

We will start with the basic system

$$G(s) = C(sI - A)^{-1}B = C\Phi B. \tag{9.22}$$

where, as before,

$$\Phi = (sI - A)^{-1}. \tag{9.23}$$

Without specifying the controller or observer gain matrices, we can represent the full state closed loop systems and the observer-based closed loop system with the state space form; i.e.

Full State:

$$\begin{cases} \dot{x} = Ax + Bu \\ u = -Kx \end{cases} \tag{9.24}$$

Observer-based:

$$\begin{cases} \dot{x} = Ax + Bu \\ \dot{\hat{x}} = A\hat{x} + Bu + F(y - C\hat{x}) = (A - FC)\hat{x} + Bu + Fy \\ u = -K\hat{x} \end{cases} \tag{9.25}$$

Taking the Laplace transform of (9.24) and (9.25) would result in

$$x(s) = \Phi B u(s) \tag{9.26}$$

$$(sI - A)\hat{x}(s) = Bu(s) + Fy(s) - FC\hat{x} = -(BK + FC)\hat{x}(s) + Fy(s) \tag{9.27}$$

From now on, for brevity, we will use y instead of $y(s)$, etc. The dependence of y, u etc on s will be clear from the context. We can now make a few interesting remarks:

1. Note that the compensator may not be stable!

$$u = -K(\Phi^{-1} + BK + FC)^{-1}Fy = -K\Psi Fy \tag{9.28}$$

where by Ψ we clearly mean

$$\Psi = (sI - A + BK + FC)^{-1}. \tag{9.29}$$

2. We may also consider the LQR controller as a controller designed for general purpose (and not necessarily regulation). So consider Figures 9.1 and 9.2, where the closed loop system is presented in the block diagram form, with potentially nonzero reference input r. Keep in mind that LQR/LQG are designed for the regulation problem, but for other performance specifications (robustness, noise property, etc), the overall shape of the loop might be of some interested. Now, note that the transfer function from the reference to input is the same in full

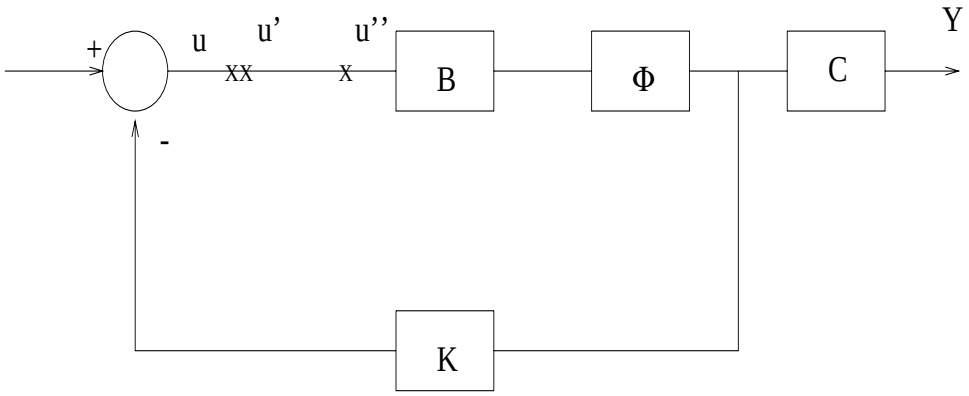

Figure 9.1: Block diagram for the full state controller

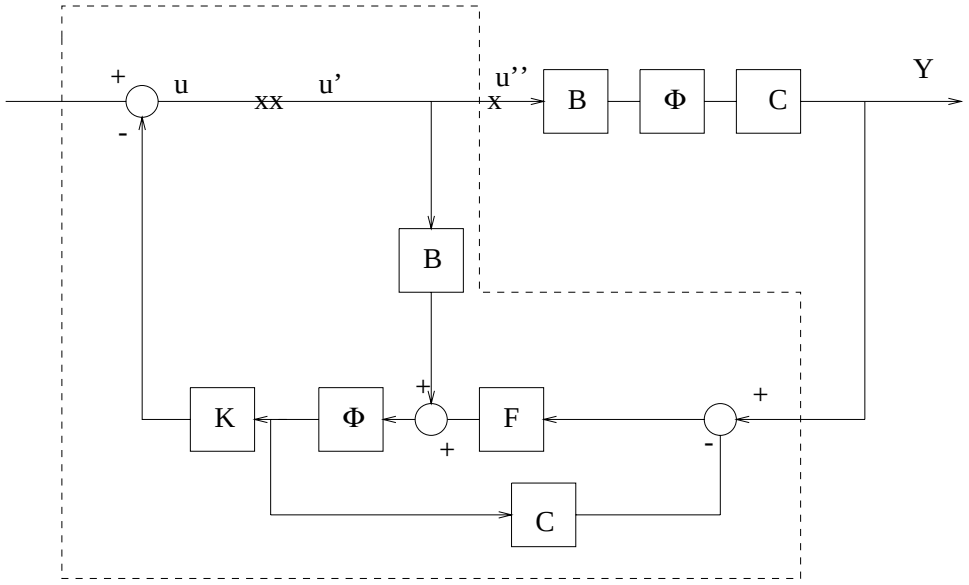

Figure 9.2: Block diagram for the observer based controller

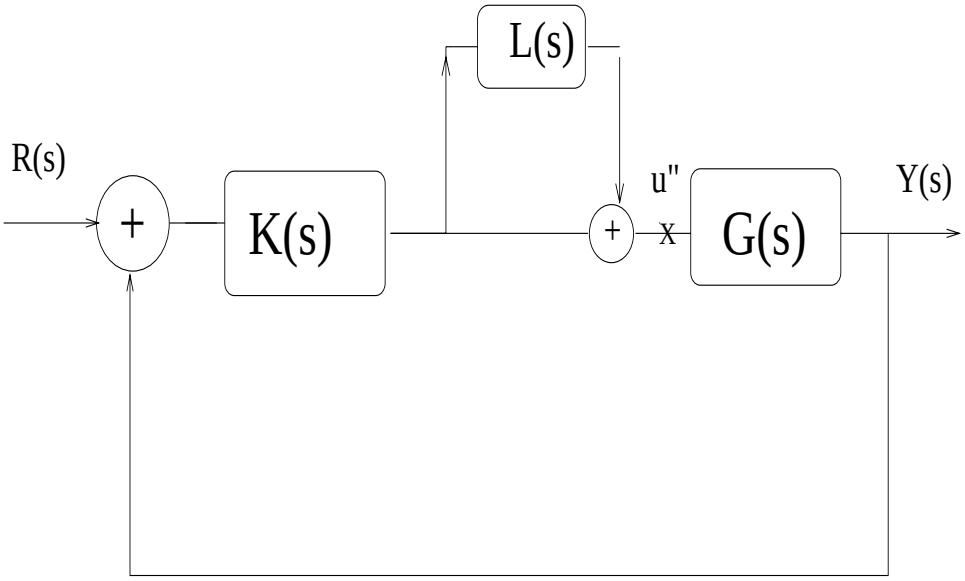

Figure 9.3: Multiplicative uncertainty at the input

state (Figure 9.1) and observer based controllers (Figure 9.2). For the full state case

$$u = r - Kx = r - K\Phi Bu \rightarrow u = (I + K\Phi B)^{-1} r \rightarrow x = \Phi B(I + K\Phi B)^{-1} r \quad (9.30)$$

while for the observer

$$(sI - A + FC)\hat{x} = Bu + FCx = Bu + FC\Phi Bu$$

or

$$(sI - A + FC)\hat{x} = (\Phi^{-1} + FC)\Phi Bu \longrightarrow \hat{x} = \Phi Bu \quad (9.31)$$

the rest follows exactly as the full state case to obtain

$$\hat{x} = \Phi B(I + K\Phi B)^{-1} r. \quad (9.32)$$

From now on, we will focus of the regulator problem and thus $r = 0$. Also, consider Figures 9.1 and 9.2, assume that we will consider 'breaking' the loop at two places; X and XX (the reason for this will become obvious below). Therefore we can consider u the output of the compensator and u' and u'' the input in the plant, depending on where the loop is broken.

Let us discuss the motivation for this. Consider the block diagram of Figure 9.3. The configuration is called 'multiplicative uncertainty at the input', and results in plants of the form $G(I + L)$. Recall that during the compensator design, only the nominal system is used. Information on $L(jw)$, however, is

used to design a compensator that accommodates this uncertainty (e.g., the loop shape for KG or GK). Clearly, u and u'' are not the same signal. The first is the output of the compensator and the second is the actual input to the plant (that is u after going through the uncertain dynamics, hence the name of uncertainty at the input!). Furthermore, regardless of the form of controller, the transfer function from u'' to u is exactly KG! (check this yourself). If robustness with respect to uncertainty is your concern, you better make sure the KG (or GK, if appropriate) of the final compensator satisfies the requirements.

Now we can continue with our rambling.

3. In both Figures 9.1 and 9.2, the transfer function from u' to u is $-K\Phi B$. (To see this better, assume that there are unmodeled junk between u and u'. The actual output to the plant is therefore u', while the output of the compensator is u).

$$FullState: \quad u = -Kx = -K\Phi Bu'$$

$$Observer: \quad u = -K\hat{x} \stackrel{?}{=} -K\Phi Bu'$$

but u' in this case is the plant input and similar to (9.31), we can obtain $\hat{x} = \Phi Bu'$. The rest is obvious.

This fact is not particularly interesting! After all, considering Figure 9.2, the point u' is still *inside* the compensator box! A more reasonable way of incorporating unmodeled dynamics is to break the loop outside of the compensator (point X) and consider the transfer function from u'' to u for both controllers.

For the full state feedback, we still have $x = \Phi Bu''$, therefore

$$u = -K\Phi Bu''. \tag{9.33}$$

For the observe, however, things get rather complicated

$$\hat{x} = \Phi Bu + \Phi FC\Phi Bu'' - \Phi FC\hat{x} \tag{9.34}$$

taking into account the control law in (9.25)

$$\hat{x} = (I + \Phi FC + \Phi BK)^{-1}\Phi FC\Phi Bu'' \tag{9.35}$$

or equivalently

$$u = -K(I + \Phi FC + \Phi BK)^{-1}\Phi FC\Phi Bu''. \tag{9.36}$$

Considering the fact that $C\Phi B$ is the plant (i.e., $G(s)$) and u'' the input, we see that $C\Phi Bu''$ is nothing but the output. Equation (9.36) is therefore of the form $u = P(s)y$, where P is the compensator. In summary, the transfer function from u'' to u in (9.36) is the 'KG' of the observer based controller! (which, by now, should not surprise you).

Equation (9.36), in general, is not the same as (9.33). To explore the possibility of making them the same, let us rewrite (9.35). We start by rearranging (9.34)

$$(\Phi^{-1} + FC)\hat{x} = Bu + FC\Phi Bu''$$

or

$$\hat{x} = (\Phi^{-1} + FC)^{-1}[Bu + FC\Phi Bu''] \tag{9.37}$$

but, from the matrix inversion lemma, we have

$$(\Phi^{-1} + FC)^{-1} = \Phi - \Phi F(I + C\Phi F)^{-1}C\Phi. \tag{9.38}$$

Simple manipulations show that

$$[\Phi - \Phi F(I + C\Phi F)^{-1}C\Phi]B = \Phi[B - F(I + C\Phi F)^{-1}C\Phi B] \tag{9.39}$$

and

$$\begin{aligned}[\Phi - \Phi F(I + C\Phi F)^{-1}C\Phi]FC\Phi B &= \Phi F[I - (I + C\Phi F)^{-1}C\Phi F]C\Phi B \\ &= \Phi F(I + C\Phi F)^{-1}C\Phi B. \end{aligned} \tag{9.40}$$

HERE COMES THE LIGHTENING! **If** we had

$$F(I + C\Phi F)^{-1} = B(C\Phi B)^{-1} \tag{9.41}$$

from (9.37) to (9.40), the transfer function from u'' to u, via (9.37) would become

$$u = -K\Phi Bu'' \quad !\,!\,!$$

that is , the same as the full state feedback. Choosing F such that (9.41) holds is the central idea of the LTR method. This is not as easy as you might think! To see the difficulty, consider the following. Equation (9.41) implies that

$$\Phi F(I + C\Phi F)^{-1}C\Phi B = \Phi B. \tag{9.42}$$

However, from matrix inversion lemma, we can write

$$\Phi F(I + C\Phi F)^{-1}C\Phi = \Phi - (\Phi^{-1} + FC)^{-1}.$$

For (9.42) to hold, in light of the previous equation, we need

$$(\Phi^{-1} + FC)^{-1}B = 0 \ \ \forall\ w. \longrightarrow (jwI - A + FC)^{-1}B = 0 \ \ \forall w,$$

which is not an easy condition to satisfy. The traditional approaches to this problem end up having observer gains that get larger and larger and satisfy this condition only *asymptotically.*

9.4.2 Observer Design

Recall (9.41). We would like to design an observer such that the gain satisfies

$$F(I + C\Phi F)^{-1} = B(C\Phi B)^{-1}.$$

One way to satisfy this is to have $F = F_q$ (i.e., F a function of scalar q) such that

$$\frac{F_q}{q} \longrightarrow BW \quad , \quad W \; nonsigular. \tag{9.43}$$

In that case

$$F(I + C\Phi F)^{-1} = \frac{F_q}{q} q(I + C\Phi F_q)^{-1} = \frac{F_q}{q}(\frac{I}{q} + C\Phi\frac{F_q}{q})^{-1}$$

and as $q \longrightarrow \infty$, we get

$$F(I + C\Phi F)^{-1} \to BW(C\Phi BW)^{-1} = B(C\Phi B)^{-1}$$

which is (9.41). Therefore, instead of (9.41), we can focus on (9.43). Now consider the Kalman filter equation

$$AS + SA^T + Q_f - SC^T R^{-1} CS = 0, \quad F = SC^T R^{-1} \tag{9.44}$$

Instead of a typical Q_o, use

$$Q_f = Q_o + q^2 BVB^T \tag{9.45}$$

where V can be any positive definite matrix (choose I if you want). The Riccati equation in (9.44) can be written as

$$A\frac{S}{q^2} + \frac{S}{q^2}A^T - q^2\frac{S}{q^2}C^T R^{-1} C\frac{S}{q^2} + BVB^T + \frac{Q_o}{q^2} = 0. \tag{9.46}$$

From basic properties of Riccati equation, **if the nominal system is minimum-phase and left invertible**, then

$$q \to \infty \implies \frac{S}{q^2} \to 0 \tag{9.47}$$

and furthermore, along the lines of equation (9.8)

$$F = SC^T R^{-1} \to qBV^{\frac{1}{2}} U R^{-\frac{1}{2}} = qBW \tag{9.48}$$

for some nonsingular W, as desired. So the method boils down to the following easy to use, NASA approved bullet chart

- Design the LQR such that KG has the appropriate loop shape
- Check loop shape after adding the nominal observer

- Increase q in (9.45) and (9.46), to recover the LRQ loop, pointwise in the frequency domain

Almost everything here can be done with the dual problem. When uncertainty (and disturbances) are best modeled by multiplicative uncertainty at the output, the approach is to design the filter so that GK has the desired shape. Then by manipulating the control Riccati equation recover the loop. We can now present the last bullet chart regarding the short comings of the LTR process

- Requires minimum phase property
- Uncertainty either at the input or at the output
- Can only recover the loop through high gains (i.e., asymptotically)

9.5 PROBLEM SET

Exercise 9.3. *Make sure you can follow the steps from* (9.5) *and* (9.6)*. Also, why is the right hand side of* (9.6) *greater than or equal to the identity matrix for* $s = jw$*, for all* w*?*

Exercise 9.4. *show how* (9.6) *implies that* (9.12) *holds.*

Exercise 9.5. *The conditions in* (9.9) *are claimed to cause any loss of generality. Can it be true?*

Exercise 9.6. *Show i:* (9.28) *holds, ii:* (9.39) *and* (9.40) *hold, and iii:* (9.48) *holds.*

Exercise 9.7. *Is the following identity true?*

$$(sI - A + FC)^{-1} = [I + (sI - A)^{-1}FC]^{-1}\ (sI - A)^{-1}$$

Exercise 9.8. *Recall that we had* $K_{LQG} = -K(sI - A + BK + FC)^{-1}F$*. Show that this is the same as*

$$K_{LQG} = [I + K(sI - A + FC)^{-1}B]^{-1}K(sI - A + FC)^{-1}F$$

10 Numerical Linear Algebra and Control

10.1 LMI

A Linear Matrix Inequality (LMI) has the following form:

$$F(x) = F_o + \sum_{i=1}^{m} x_i F_i \ > 0 \tag{10.1}$$

where $x \in \mathcal{R}^m$ is the unknown variables and F_i for $i = o, 1, \ldots m$ are known symmetric matrices. This whole chapter is due to the fast that now we have efficient, easy to use , software that can solve for x, once F_i are given, basically because once the variable enters linearly of affinely, the search is convex. The software is actually available via the LMI ToolBox of Matlab.

While a course in optimization is the proper place to study this, for now all we need are a couple of facts:

- If the variables enter linearly, the search is convex
- If the search is convex, LMI tool box will find a solution if one exists

We can have matrices in (10.1), as everything will follow - as long as the unknown matrix variables enter linearly. For example, consider the search for $P > 0$, such that $-T = PA + A^T P < 0$ (for simplicity, let us do the case of dimension 2)

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = p_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} > 0$$

or using a notation similar to (10.1)

$$P = p_1 E_1 + p_2 E_2 + p_2 E_3 > 0 \tag{10.2}$$

where the notation is obvious. Now we can write

$$PA + A^T P = p_1(A^T E_1 + E_1 A) + p_2(A^T E_2 + E_2 A) + p_3(A^T E_3 + E_3 A) < 0$$

combining the two conditions (on P and $PA + A^T P$), we get the following

$$\sum_{i=1}^{3} p_i \begin{bmatrix} -A^T E_i - E_i A & 0 \\ 0 & +E_i \end{bmatrix} > 0$$

which clearly is in the form of (10.1) - without F_0! Clearly, as long as you have linear appearance in the unknown matrix or scalar variables, we can do this trick. Note how we combined 2 different conditions into 1 (i.e., several LMI's in which the unknown appears linearly or affinely is still a convex problem). Fortunately, this kind of tedious manipulation is done by Matlab!

A related problem is the following

$$\text{minimize } c^T x \text{ subject to } F(x) > 0 \tag{10.3}$$

where c^T is the selection vector and x is the vector of unknowns. This is also a problem Matlab solves (though the 'mincx' function), though you probably can think of a way to solve it iteratively too.

The availability of LMI tool box (due to much progress in numerical linear algebra) has led to tremendous progress in control design - which is the main reason we have this chapter! Roughly speaking, we study different analysis and synthesis problem and manipulate things enough until they are in the form of either (10.1) or (10.3). Then we declare the problem solved (if there is a solution, Matlab will solve it).

10.2 Schur Complement

Let

$$A = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \tag{10.4}$$

Then, the following are equivalent:

- $$A > 0 \Longleftrightarrow \begin{cases} Q > 0 \\ R > 0 \\ Q - SR^{-1}S^T > 0 \end{cases}$$
- $$A > 0 \Longleftrightarrow \begin{cases} Q > 0 \\ R > 0 \\ R - S^TQ^{-1}S > 0 \end{cases}$$

Proof: Clearly, $A > 0$ implies $Q > 0$ and $R > 0$ (use $\begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ y \end{pmatrix}$ in the basic definition of positive definiteness). So we concentrate on the third property; i.e., given $Q > 0$ and $R > 0$, then $A > 0$ is equivalent to the third property. Obviously, we only need to do the first form (and the second follows closely).

We start with a generic vector $z = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$z^T A z = x^T Q x + y^T R y + x^T S y + y^T S^T x > 0 \quad \forall x, y \tag{10.5}$$

To show $\Rightarrow$, we use *any* $x \neq 0$ along with $y = -R^{-1}S^T x$ to get

$$\begin{aligned} z^T A z &= x^T Q x + x^T S R^{-1} R R^{-1} S^T x - x^T S R^{-1} S^T x - x^T S R^{-1} S^T x \\ &= x^T (Q - S R^{-1} S^T) x > 0 \quad \forall x \neq 0 \end{aligned}$$

thus $A > 0$ implies that $(Q - SR^{-1}S^T) > 0$.

To show $\Leftarrow$, we go back to equation (10.5) : Add and subtract a $x^T S R^{-1} S^T x$, to get :

$$\begin{aligned} z^T A z &= x^T Q X - x^T S R^{-1} S^T x + x^T S R^{-1} S^T x + y^T R y + x^T S y + y^T S^T x \\ &= x^T (Q - S R^{-1} S^T) x + (y^T + x^T S R^{-1})\ R\ (y + R^{-1} S^T x) \end{aligned}$$

for all x and y. The first term is strictly positive and the second term is in the form of $B^T R B$ with $R > 0$, so it is as worst zero. Therefore $z^T A z > 0 \ \ \forall z$. $\square$

What if we had $A \geq 0$? The only modification would have been the following:

- If $R > 0$, then
$$A \geq 0 \Longleftrightarrow \left\{ \begin{array}{l} Q \geq 0 \\ Q - S R^{-1} S^T > 0 \end{array} \right. \tag{10.6}$$
- If $Q > 0$, then
$$A > 0 \Longleftrightarrow \left\{ \begin{array}{l} R \geq 0 \\ R - S^T Q^{-1} S > 0 \end{array} \right. \tag{10.7}$$

Naturally, everything above follows if we replace all '>' signs with '<' signs. That is

- $$A < 0 \Longleftrightarrow \left\{ \begin{array}{l} Q < 0 \\ R < 0 \\ Q - S R^{-1} S^T < 0 \end{array} \right. \tag{10.8}$$
- $$A < 0 \Longleftrightarrow \left\{ \begin{array}{l} Q < 0 \\ R < 0 \\ R - S^T Q^{-1} S < 0 \end{array} \right. \tag{10.9}$$

Finally, we have the following: IF

$$C = \left(\begin{array}{cc} Q & S \\ S^T & 0 \end{array} \right) \geq 0 \ \ (or \leq 0) \ \ then \ \ S = 0 \tag{10.10}$$

(pick any $z^T = (x^T \ y^T)$, show that there exists another vector $\hat{z}^T = (x^T \ \hat{y}^T)$ such that the sign of $z^T C z$ is opposite of $\hat{z}^T C \hat{z}$ which is impossible. The only way out is to have $x^T S y = 0$ for all x and y which is mean $S = 0$)

10.3 S-Procedure

Let T_o, T_1, $\cdots$, T_p be symmetric matrices. If there exists a set of positive scalars $\tau_i > 0$ such than

$$T_o - \sum_1^p \tau_i T_i > 0 \tag{10.11}$$

then the following is true

$$x^T T_o x > 0 \quad for\ all \quad x \neq 0\ such\ that\ x^T T_i x > 0\ \forall\ i \tag{10.12}$$

Clearly, (10.11) implies (10.12) - to see this just multiply (10.11) by x^T and x. We end up using it as a sufficient condition - possibly conservative - by finding τ_i such that (10.11) holds. When $p = 1$, this is sufficient and necessary, but in general there is some conservatism associated with it.

We often use negative conditions, which are obtained by using$S_i = -T_i$: Let S_o, S_1, $\cdots$, S_p be symmetric matrices. If there exists a set of positive scalars $\tau_i > 0$ such than

$$S_o - \sum_1^p \tau_i S_i < 0 \tag{10.13}$$

then the following is true

$$x^T S_o x < 0 \quad for\ all \quad x \neq 0\ such\ that \quad x^T S_i x < 0\ \forall\ i \tag{10.14}$$

10.4 Ellipsoids

We will use the concept of ellipsoidal sets a great deal. Simply, consider the following: Given $P > 0$, we define the following ellipsoid:

$$\mathcal{E}(P, c) = \{x : x^T P x \leq c\ \} \tag{10.15}$$

Naturally, we need $P > 0$ if we want a bounded set. Let us consider some of its basic properties:

- $x \in \mathcal{E}(P, c) \quad \Leftrightarrow \quad \alpha x \in \mathcal{E}(P, c\alpha^2)$
- $c_1 \leq c_2 \quad \Leftrightarrow \quad \mathcal{E}(P, c_1) \subseteq \mathcal{E}(P, c_2)$
- If $P_1 \leq P_2 \quad \Leftrightarrow \quad \mathcal{E}(P_2, c) \subseteq \mathcal{E}(P_1, c)$
- ellipsoid not centered at the origin:

$$\mathcal{E}(P, x_o, c) = \{x : (x - x_o)^T P (x - x_o) \leq c\ \}$$

- The volume of an ellipse is $\alpha_n(detP^{-1})^{\frac{1}{2}}$, where α is the volume of the n-dimensional unit 2-ball; i.e.,

$$\alpha_n = \frac{\pi^{n/2}}{(n/2)!} \; if \; n \; is \; even$$

$$\alpha_n = \frac{2^n \pi^{(n-1)/2}((n-1)/2)!}{n!} \; if \; n \; is \; odd$$

but an upper bound (thus only a potentially conservative estimate) to it is more suitable for matlab use. For this, recall (??)

$$(detP)^{1/n} \leq \frac{trace(P)}{n}$$

which implies that to minimize volume (thus $det(P)$ - which is very difficult), we can try to minimize $trace(P^{-1})$ or at times $trace(Q)$ where $Q > P^{-1}$.

10.4.1 Norm of a vector in an ellipsoid

One of the main tricks we will be using, over and over, is to find the max of $\|y = Cx\|$ for all $x \in \mathcal{E}(P, c)$ (what is the minimum value?). Consider the following

$$\begin{pmatrix} P & C^T \\ C & \frac{\gamma^2}{c} I \end{pmatrix} > 0 \Leftrightarrow \quad P - \frac{c}{\gamma^2} C^T C > 0 \tag{10.16}$$

Therefore, if this condition (i.e., (10.16)) holds, we have

$$\|y\| \leq \gamma$$

10.5 Stability, L_2 gain, etc

We now start with the analysis results; i.e., conditions that can be checked relatively easily and would imply stability, finite gain, etc.

10.5.1 stability

Suppose we have the following dynamics

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_o \end{cases} \tag{10.17}$$

One way of checking the stability of A was through Lyapunov method and Lyapunov equation: finding the solution

$$PA + A^T P = -Q \quad , \quad Q > 0$$

which means we must have $P > 0$ such that $PA + A^T P < 0$. This is a simple search (for P) which is convex, this easily done with Matlab LMI software.

10.5.2 H_∞ or L_2 gain

Suppose we have the following dynamics

$$\begin{cases} \dot{x} & = Ax + B_1 w \\ y & = Cx + Dw \\ x(0) & = 0 \end{cases} \tag{10.18}$$

We want to find the L_2 or energy gain of the system, ie., the smallest γ such that

$$\int_0^\infty y^T y \; dt \leq \gamma^2 \int_0^\infty w^T w \; dt$$

(this also becomes the famous H_∞ problems). The basic approach is to use a Lyaponuv like function

$$V(x) = x^T(t) P x(t)$$

where $P > 0$, and find the smallest γ that satisfies:

$$\mathcal{T} = \dot{V}(t) + y^T y - \gamma^2 w^T w \leq 0 \tag{10.19}$$

since simply integrating both sides from zero to infinity gives the energy (or L_2) gain. Note that this gives an estimate (or upper bound) on the actual gain and there may be some conservatism.

To get a more tractable form of (10.19), we simply take derivative of V and substitute:

$$\begin{aligned} \mathcal{T} = \dot{V}(t) + y^T y - \gamma^2 w^T w = & x^T PAx + x^T A^T Px + x^T PBw + w^T B^T Px \\ & (Cx + Dw)^T (Cx + Dw) - \gamma^2 w^T w \end{aligned}$$

which can be written as

$$\mathcal{T} = \begin{pmatrix} x^T & w^T \end{pmatrix} \begin{pmatrix} PA + A^T P + C^T C & PB + C^T D \\ B^T P + D^T C & -\gamma^2 I + D^T D \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

A sufficient condition for (10.19) is this the following

$$\begin{pmatrix} PA + A^T P + C^T C & PB + C^T D \\ B^T P + D^T C & -\gamma^2 I + D^T D \end{pmatrix} < 0 \tag{10.20}$$

which is the same as the following (doing the Schur complement)

$$\begin{pmatrix} PA + A^T P & PB \\ B^T P & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} C^T \\ D^T \end{pmatrix} I \begin{pmatrix} C & D \end{pmatrix} < 0$$

but using Schur complement formula, this is equivalent to

$$\begin{pmatrix} PA + A^T P & PB & C^T \\ B^T P & -\gamma^2 I & D^T \\ C & D & -I \end{pmatrix} < 0. \tag{10.21}$$

A sequence of pre and post multiplying by $\begin{pmatrix} 1/\sqrt{\gamma} & 0 & 0 \\ 0 & 1/\sqrt{\gamma} & 0 \\ 0 & 0 & \sqrt{\gamma} \end{pmatrix}$, and using $\hat{P} = P/\gamma$ yields

$$\begin{pmatrix} \hat{P}A + A^T\hat{P} & \hat{P}B & C^T \\ B^T\hat{P} & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0. \tag{10.22}$$

There is another - equivalent - form that is used in the synthesis problems. That form is obtained by pre and post multiplying (10.22) for example, by $\begin{pmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ where $Q = \hat{P}^{-1}$, results in

$$\begin{pmatrix} AQ + QA^T & B & QC^T \\ B^T & -\gamma I & D^T \\ CQ & D & -\gamma I \end{pmatrix} < 0. \tag{10.23}$$

Often we see references to Bounded Real Inequality, which could be any of the above. In particular, (10.20) can be written (doing a Schur complement) as

$$PA + A^TP + C^TC + (PB + C^TD)(\gamma^2 - D^TD)^{-1}(B^TP + D^TC) < 0 \tag{10.24}$$

or if $D = 0$, the simple form of

$$PA + A^TP + C^TC + \frac{1}{\gamma^2}PBB^TP < 0. \tag{10.25}$$

In all of these problems (i.e., (10.20)-(10.23)), we seek a positive definite matrix (P or Q) so that the linear matrix inequality - in which the variable P enters linearly- is satisfied. This is a standard convex search and LMI toolbox can be used to solve for it. Furthermore, we would like to minimize γ. Again, this ends up being a convex problem (generalized eigenvalue problem) for which matlab can be used easily (if nothing else, think of matlab solving this minimization problem as a sequence of feasibility problems with decreasing values of γ which should converge to any tolerance in finite steps).

10.5.3 Invariant/Reachable sets

Consider the system in (10.18). Let us try to find some estimates for the reachable set or the invariant set. The reachable set (from zer) is the set of points the state vector can reach with zero initial condition, given some limitations on the disturbance. The invariant set is a similar (but not identical necessarily) concept. The invariant set is the set that the state vector does not leave once it is inside of it, again given some limits on the disturbance. (which one of then is stronger, so to speak, why?). Generally, these are hard to characterize (as you might have seen in the exam before!).

We discuss both types, depending on the form of the disturbance bound. In all case, we use a Lyapunov function and an ellipsoidal sets for this estimate; i.e., our estimates are in the following form

$$\mathcal{E} = \{x : V(x) = x^T P x \leq c\}$$

Obviously, there are varying degrees of conservatism associated with these, but as you will see, they are easy to obtain and can be applied to nonlinear systems as well.

Case A:Energy Bounded Disturbance:

Suppose the disturbance w had a energy bound;

$$\int_0^\infty w^T w \; dt \leq w_{max}^2$$

then if we can find P such than

$$\dot{V} - w^T w \leq 0 \tag{10.26}$$

then the reachable set is $V(x) \leq w_{max}^2$. To see this, simply integrate (10.26) from zero to t

$$V(t) - V(0) - \int_0^t w^T w \; dt \leq 0$$

With zero initial conditions, we have

$$V(x) = x^T P x \leq \int_0^t w^T w \; dt < w_{max}^2$$

which can also be refined as

$$\|x\|^2 < \frac{1}{\lambda_{min} P} w_{max}^2.$$

In any case, to find the V or P, we expand the $\dot{V}$ term in (10.26); i.e., we seek P such than the following holds

$$x^T P A x + x^T A^T P x + x^T P B w + w^T B^T P x - w^T w < 0$$

where this is the same as

$$\begin{pmatrix} x^T & w^T \end{pmatrix} \begin{pmatrix} PA + A^T P & PB \\ B^T P & -I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} < 0$$

As a result, we have the following estimate (sufficient condition): Suppose there is $P > 0$ satisfying

$$\begin{pmatrix} PA + A^T P & PB \\ B^T P & -I \end{pmatrix} < 0 \tag{10.27}$$

then an estimate of the reachable set is

$$\mathcal{E} = \{x : x^T P x \leq w_{max}^2\}$$

Case B: Peak Bounded Disturbance:

Suppose we have a bound on the peak norm of the disturbance:

$$w^T(t)w(t) \leq w_{max}^2 \quad \forall t$$

Note that if you are interested in element wise peak norm, you might be off by a factor of $\sqrt{m}$ where m is the dimension of w. Here the basic condition is the following: Suppose there was a P for a $V = x^T P c$ such that

$$\dot{V} + \alpha(V - w^T w) < 0 \tag{10.28}$$

for some $\alpha > 0$. Then $V \leq w_{max}^2$ is an attractive invariant set (i.e., if you start from inside, you never leave, and if you start from outside, you get attracted to it!). It is not that important perhaps, but α come from the S-procedure (setting the problem up as having $\dot{V} < 0$ where $V > w^T w$). It is relatively easy to see that (10.28) implies that

$$\dot{V} + \alpha(V - w_{max}^2) < 0$$

which implies that if you are insider of $V \leq w_{max}^2$ you cannot go outside and if you are outside, V gets smaller - until you get inside! Going through the same calculations as before we get the following conditions for P:

$$\begin{pmatrix} PA + A^T P + \alpha P & PB \\ B^T P & -\alpha I \end{pmatrix} < 0 \tag{10.29}$$

then an estimate of the reachable (and/or invariant) set is

$$\mathcal{E} = \{x : x^T P x \leq w_{max}^2\}$$

Note that in this case, the term αP is nonlinear in unknown variables, which destroys the convexity. Generally, a simple line search (ie, iterative) is done on α. This is not a major problem, since the (1,1) block of the inequality above shows that α is between zero and half of the real part of the least stable eigenvalue of A (why?).

10.5.4 Energy to peak and peak to peak gains

To obtain (upper) bounds for the energy-to-peak or peak-to-peak gains for a system, we simply combine the previous subsection results with those regarding the norm of a vector in the ellipsoid:

- **Energy to Peak norm**: Suppose (10.27) and (10.16) hold for some P, $\gamma = \gamma^*$ and $c = 1$, then it is easy to see that $x^T P x \leq 1$ is the reachable set, as long as $\|w\|_{L_2} \leq 1$. Then, (10.16) implies that the norm of y is less than γ^* in this ellipsoid.

- **Peak to Peak norm**: Suppose (10.29) and (10.16) hold for some P, $\alpha > 0$, $\gamma = \gamma^*$ and $c = 1$, then it is easy to see that $x^T P x \leq 1$ is the reachable set, as long as $\|w(t)\| \leq 1$. Then, (10.16) implies that the norm of y is less than γ^* in this ellipsoid.

As a result, we solve for $P > 0$, such that the LMI's holds, while minimizing γ^*.

10.6 Synthesis

Suppose our system is in the follwoing form

$$\dot{x} = Ax + B_1 w + B_2 w \tag{10.30}$$
$$z = C_1 x + D_{11} w + D_{12} u \tag{10.31}$$
$$y = C_2 x + D_{21} w + D_{22} u \tag{10.32}$$

where w is the disturbance and u is the control input. Vector z is called controlled output, and contains states or combinations that we want to penalized or reduce, etc (which may be the same or different from the measured output y). Almost always, we use $D_{22} = 0$, to simplify things (well posed problem issue). Often it is justified through a simple transformation of the form $\hat{y} = y - D_{22} u$ - as long as u is available.

Throughout this long subsection, we will try the synthesis problem: i.e, finding a control law - with different structures - such that the closed loop has desirable properties (e.g., stability, small L_2 gain, small peak to peak gain, etc).

10.6.1 State Feedback Controllers

Suppose you are designing a state feedback controller of the form

$$u = Kx \tag{10.33}$$

Putting it back in the original equation of motion (i.e., (10.30)), yields the following closed loop equation:

$$\dot{x} = (A + B_2 K)x + B_1 w \tag{10.34}$$
$$z = (C_1 + D_{12} K)x + D_{11} w \tag{10.35}$$

We try to do the problem of stabilizing controller only; i.e., when $w = 0$. Other considerations, such as minimizing the L_2 gain of the closed loop or the energy to peak or peak to peak gains are quite similar, and left as exercise.

We start by hoping to find a matrix P such that a Lyapunov funcation candidate of the of form

$$V(x) = x^T P x$$

can do the trick. For this $V(x)$ to work, we need $\dot{V}(x) < 0$; i.e.

$$P(A + B_2K) + (A + B_2K)^T P = PA + PB_2K + A^T P + K^T B_2^T P < 0 \quad (10.36)$$

The problem is the PB_2K term, which is nonlinear in the unknown variables (i.e., not a linear matrix inequality, thus not a convex search and no LMI-toolbox!). Fortunately, there is a little trick that solves this dilemma. Use

$$W = KP^{-1} = KX \quad (10.37)$$

and note that pre- and post multiplying (10.36) by $X = P^{-1}$ we get the following sufficient condition

$$AX + XA^T + B_2W + W^T B_2^T < 0 \quad (10.38)$$

which is now linear in X and W. So we use LMI-toolbox (or any similar program) to find W and $X > 0$ that satisfy (10.38). Then the control law (from (10.37)) is

$$K = WX^{-1}.$$

10.6.2 Output feedback design

Now suppose we do not have access to all of states and instead had access to :

$$\begin{cases} \dot{x} &= Ax + B_1w + B_2u \\ z &= C_1x + D_{11}w + D_{12}u \\ y &= C_2x + D_{21}w \end{cases} \quad (10.39)$$

Where, as before, we have assumed that there is no feed-through term from u to y. We need to design a compensator of the form:

$$\begin{cases} \dot{x}_c &= A_cx_c + B_cy \\ u &= C_cx_c \end{cases} \quad (10.40)$$

Combining the two, we get the closed loop dynamics of

$$\begin{cases} \dot{x}_{cl} &= A_{cl}x_{cl} + B_{cl}w \\ z &= C_{cl}x_{cl} + D_{cl}w \end{cases} \quad (10.41)$$

where $x_{cl}^T = (\ x^T \ \ x_c^T)$ and

$$A_{cl} = \begin{pmatrix} A & B_2C_c \\ B_cC_2 & A_c \end{pmatrix}, \quad B_{cl} = \begin{pmatrix} B_1 \\ B_cD_{21} \end{pmatrix}$$

and

$$C_{cl} = (C_1 \ \ D_{12}C_c) \quad , \quad D_{cl} = D_{11}$$

Now let us discuss stability only (L_2 and stuff follows pretty similar to this) - so we set $w = 0$. For stability, it is sufficient to have $P >$ such that

$$PA_{cl} + A_{cl}^T P < 0$$

or using $Q = P^{-1}$, if we could find $Q > 0$ such that

$$A_{cl}Q + QA_{cl}^T < 0 \tag{10.42}$$

where the dimension of Q is $2n \times 2n$; i.e, has a structure

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{pmatrix} \tag{10.43}$$

Now we can say, without any loss of generality that Q_2 is non-singular (one can always add a little bit to it; e.g., $Q_2 = Q_2 + \epsilon I$ so that the off-diagonal terms is non-singular without changing $Q > 0$ or the overall inequality in (10.42)). Once this is done, we can show that any matrix in the form in (10.43) - with Q_2 nonsingular, can be transformed to the following:

$$TQT^T = \begin{pmatrix} X & X \\ X & S^{-1} + X \end{pmatrix}, , T = diag\{I, T_1\}$$

for some appropriately defined T_1. Naturally, T_1, X and S are functions of Q_i ! Furthermore, it is relatively easy to show that this transformation only changes the 'realization' of the compensator and nothing else. In summary, we can - without any loss of generality, say that if (10.42) has a solution Q, it has the form

$$Q = \begin{pmatrix} X & X \\ X & S^{-1} + X \end{pmatrix} \tag{10.44}$$

Next, we define

$$Y = S + X^{-1} \Longrightarrow \begin{pmatrix} X & X \\ X & S^{-1} + X \end{pmatrix}^{-1} = \begin{pmatrix} Y & -S \\ -S & S \end{pmatrix}$$

and go back to (10.42) and pre and post multiplying it (i.e., congruent transformation) by T_2 and T_3, respectively, where

$$T_2 = \begin{pmatrix} T_3 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad T_3 = \begin{pmatrix} Y & -S \\ I & 0 \end{pmatrix}$$

After a good bit of manipulations, we get the following - from (10.42):

$$\begin{pmatrix} YA + A^TY - SB_cC_2 - C_2^TB_c^TS & Y[A + B_2C_c]X - S[A_c + B_cC_2]X + A^T \\ * & AX + XA^T + B_2C_cX + XC_c^TB_2^T \end{pmatrix} < 0 \tag{10.45}$$

The inequality above does not 'look' linear, but if we use the following

$$W_c = C_cX$$
$$W_o = -SB_c$$
$$L = Y[A + B_2C_c]X - S[A_c + B_cC_2]X = YAX + YB_2W_c - SA_cX + W_oC_2X$$

we get

$$\begin{pmatrix} YA + A^TY + W_oC_2 + C_2^TW_o^T & L + A^T \\ L^T + A & AX + XA^T + B_2W_c + W_c^TB_2^T \end{pmatrix} < 0 \tag{10.46}$$

which is linear in X, Y, W_o, W_c and L. Once these are found (by LIM-toolbox!) we get the compensator from the following

$$C_c = W_cX^{-1} \tag{10.47}$$
$$B_c = -S^{-1}W_o \tag{10.48}$$
$$A_c = S^{-1}(-LX^{-1} + YA + YB_2C_c) - B_cC_2 \tag{10.49}$$

Actually, X and Y should be such $S = Y - X^{-1} > 0$ - which through Schur is equivalent to

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} > 0 \tag{10.50}$$

In summary, the compensator is obtained by searching for the unknown variables that satisfy (10.46) and (10.50) (and then A_c etc from (10.47))

10.7 Multi-objective problems

The basic idea of multi-objective approach is to design a controller such that two (or more) different objectives are met. Consider the state feedback problem we discussed earlier, the closed loop is

$$\dot{x} = (A + B_2K)x + B_1w \tag{10.51}$$
$$z = (C_1 + D_{12}K)x \tag{10.52}$$

where we have set $D_{11} = 0$. So suppose we needed to find K such that the L_2 gain from w to z was less than γ_2 while minimizing the energy to peak gain γ^* for disturbances with unit energy- again from w to z - to have bounded energy in z while minimizing peak. Following the development of earlier sections, these objectives will be satisfied if

$$\begin{pmatrix} \hat{P}A_{cl} + A_{cl}^T\hat{P} & \hat{P}B_{cl} & C_{cl}^T \\ B_{cl}^T\hat{P} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{pmatrix} < 0 \tag{10.53}$$

for the desired γ_2, while the second objective is to minimize γ^* in

$$\begin{pmatrix} PA_{cl} + A_{cl}^TP & PB_{cl} \\ B_{cl}^TP & -I \end{pmatrix} < 0 \tag{10.54}$$

$$\begin{pmatrix} P & C_{cl}^T \\ C_{cl} & (\gamma^*)^2 I \end{pmatrix} > 0 \tag{10.55}$$

Recall that in this case, an estimate of the reachable set is

$$\mathcal{E} = \{x : x^T P x \leq w_{max}^2 = 1\}$$

Now the most general result would be that the two objectives would be met by two different P, one for each objective, but as you recall each problem will result in a search for P and $W = KP^{-1}$. Since we want only one controller, we need to have $K_1 = W_1 P_1^{-1} = W_2 P_2^{-1}$ which kills any convexity we might have!

What is done is to use the same P in both objectives (which means the same W). This allows us to solve the problem but can be quite conservative. Dealing with this conservatism is an active area of research.

10.8 Time variations and mild nonlinearities

One of the advantages of using Lyapunov functions, as mentioned earlier, is that the basic approach (including reliance on LMI) can be extended to certain class of time varying and/or nonlinear systems - with relative ease. We will do a little bit of review of these things.

First, however, we need to look into the concept of **Quadratic Stability**. If a dynamical system is shown to have Lyapunoc function V that is quadratic in x such that $\dot{V} < -\epsilon \|x\|^2$, then the system is quadratically stable (**QS**) - which is stronger that asymptotic stability (e.g., we can show exponential decay). The simplest form of V - you guessed it - for this method is $V = x^T P X$ for some $P > 0$. As you recall, for linear time invariant (LTI) systems this was equivalent to traditional stability (checked with eigenvalues), except quadratic stability can be extended to more general systems.

10.8.1 Robustness and quadratic stability

Consider the simple mass, spring dashpot model

$$m\ddot{y} + c\dot{y} + ky = u$$

or

$$\begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu$$

If stiffness or damping where unknown (either constant or slowly changing). Then we can start with $V(x) = x^T P x$ and establish stability (e.g., when $u = 0$), or L_2 gain (when u is disturbance) or even design control. For than we typically mean to find P satisfying

$$PA(q) + A(q)^T P < 0$$

where $A(q)$ denotes dependence of A on the unknown parameter q - for example $k = k_{nom} + q$. Now this last inequality still forms an LMI as long as dependence

of $A(q)$ on q is linear. In particular, obtaining a P satisfying it follows if we had a P satisfying

$$PA(q_{min}) + A(q_{min})^T P < 0 \quad and \quad PA(q_{max}) + A(q_{max})^T P < 0$$

These lead to sufficient conditions for robust stability, but they could be conservative. For LTI systems, the famous μ-synthesis leads to much less conservatism, but QS method also applies if the uncertainty is time varying (though how fast is another story!)

10.8.2 Linear parameter varying systems

Linear parameter varying systems are those systems whose model is time varying, but linear and the variation is keys to a (or few) specific parameter - which is supposed to be measured on-line. There is a lot of similarities to gain scheduling (which is a whole new can of warms). A particularly interesting form of this is the quasi-lpv. For example, consider the simple pendulum model

$$\ddot{\theta} + \sin\theta = u$$

or in state space form (with $x^T = (\theta \;\; \dot{\theta})^T$), or

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{\sin x_1}{x_1} & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = A(\rho)x + Bu$$

where $\rho(x) = \frac{\sin x_1}{x_1}$, which leads to $0 \leq \rho \leq 1$. Now, if we measure $x_1 = \theta$, then we have $\rho(t)$ on-line! Note that model here has a lot of similarities with the robust problem discussed above, with one difference: in robust problem we doe not know the values of q - ever! In the q-lpv (or lpv) however, knowing the parameter can help us do a better job. For example, suppose you had limited torque for the pendulum. If you only had one K, this had to be chosen so that no matter what x, $Kx \leq u_{lim}$. However, you could different K's , as θ gets smaller, you would use a larger K and thus can be more aggressive!

The approach - i,e, finding $K(\rho)$ - is quite similar to gain scheduling and at times is called self-scheduling (it is scheduled based on its own response! and not external command). This is an area of research that has been quite active in recent years.

10.9 Exercises

1. Show the first three properties of ellipsoids are trues
2. Show (10.16) actually does bound the norm of y
3. Show that $\gamma^T - D^T D > 0 \Leftrightarrow \sigma_{max} D < \gamma$
4. Verify the equivalency of (10.21) and (10.22)
5. Consider the invariant set for the peak bounded disturbance (e.g., (10.29)). Show that if x is not insider this set, it will reach it (i.e., contractive). Estimate the rate of convergence!
6. For the peak bounded case, use S-procedure to some up with matrix inequality for the norm of $y = Cx + Dw$.
7. Beyond stability: How would you modify the Lyapunov inequality if we needed $x(t)$ to decay as least as fast as $e^{-\alpha t}$?
8. Do the estimate for peak to peak and energy to peak if the appropriate norm of $w(t)$ was bounded by w_{max} instead of 1.
9. In (10.30, what would be z if we wanted to penalize absolute acceleration of a single mass-spring dashpot plus an actuator?
10. In the state feedback problem, do the L_2 problem: Find $u = Kx$ such that the closed loop L_2 gain is minimized.
11. In the output feedback case, show the details needed to establish that the special structure of Q is without loss of generality - including the change of controller representation.
12. I the output feedback case, do the L_2 gain minimization problem !
13. Filling some intermediate steps: mincx vs trial and error: solve the minimum γ in one of the minimization problem by 'mincx' function and then get the same by checking feasibility with decreasing values of γ
14. **Important**: In the multi-objective problem, solve the following: Given w with energy w_{max}, minimize the L_2 gain subject to the saturation bound u_{lim}