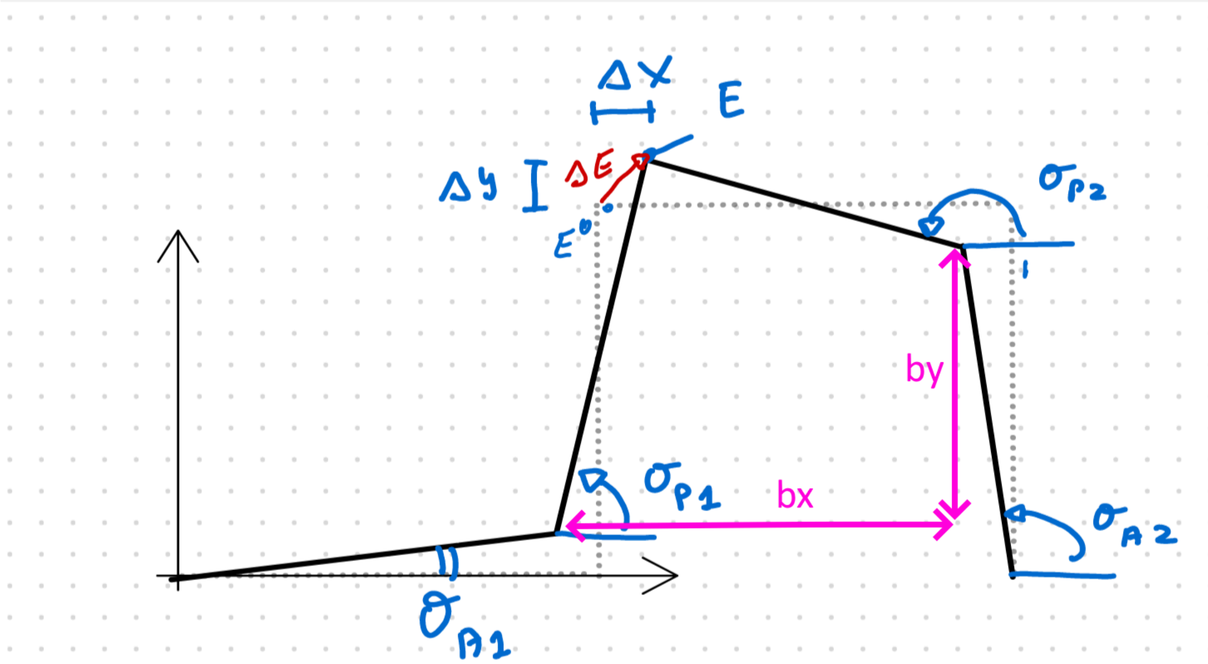
Return of the Jacobian

1) Kinematic Equations

(Angle conventions taken from “A Method for Optimal Kinematic Design of Five-bar Planar Parallel Manipulators”)



The kinematic chain in polar form is written as:

$$Le^{j\theta_{a1}} + Le^{j\theta_{p1}} = 2L + Le^{j\theta_{a2}} + Le^{j\theta_{p2}} \tag{1.}$$

Deriving in time and dividing by L:

$$\dot{\theta}_{a1} e^{j(\theta_{a1}+\frac{\pi}{2})} + \dot{\theta}_{p1} e^{j(\theta_{p1}+\frac{\pi}{2})} = \dot{\theta}_{a2} e^{j(\theta_{a2}+\frac{\pi}{2})} + \dot{\theta}_{p2} e^{j(\theta_{p2}+\frac{\pi}{2})} \tag{2.}$$

In Cartesian form:

$$\begin{cases} \dot{\theta}_{a1}\sin(\theta_{a1}) + \dot{\theta}_{p1}\sin(\theta_{p1}) = \dot{\theta}_{a2}\sin(\theta_{a2}) + \dot{\theta}_{p2}\sin(\theta_{p2}) \\ \dot{\theta}_{a1}\cos(\theta_{a1}) + \dot{\theta}_{p1}\cos(\theta_{p1}) = \dot{\theta}_{a2}\cos(\theta_{a2}) + \dot{\theta}_{p2}\cos(\theta_{p2}) \end{cases} \tag{3.}$$

2) Rotational Singularities

Let’s define

$$\omega_r = \begin{bmatrix} \dot{\theta}_{p1} \\ \dot{\theta}_{p2} \end{bmatrix}, q = \begin{bmatrix} \dot{\theta}_{a1} \\ \dot{\theta}_{a2} \end{bmatrix} \tag{4.}$$

We want to find a form like

$$J_r \omega_r = J_q q \tag{5.}$$

(taken from “Inverse kinematics and singularity analysis of a redundant parallel robot “)

We can rearrange to have all the passive and active joint angles on one side:

$$\begin{cases} \dot{\theta}_{a1}\sin(\theta_{a1}) - \dot{\theta}_{a2}\sin(\theta_{a2}) = -\dot{\theta}_{p1}\sin(\theta_{p1}) + \dot{\theta}_{p2}\sin(\theta_{p2}) \\ \dot{\theta}_{a1}\cos(\theta_{a1}) - \dot{\theta}_{a2}\cos(\theta_{a2}) = -\dot{\theta}_{p1}\cos(\theta_{p1}) + \dot{\theta}_{p2}\cos(\dot{\theta}_{p2}) \end{cases} \quad 6.$$

Convert it in matrix form

$$\begin{bmatrix} -\sin(\theta_{p1}) & \sin(\theta_{p2}) \\ -\cos(\theta_{p1}) & \cos(\theta_{p2}) \end{bmatrix} \begin{bmatrix} \dot{\theta}_{p1} \\ \dot{\theta}_{p2} \end{bmatrix} = \begin{bmatrix} \sin(\theta_{a1}) & -\sin(\theta_{a2}) \\ \cos(\theta_{a1}) & -\cos(\theta_{a2}) \end{bmatrix} \begin{bmatrix} \dot{\theta}_{a1} \\ \dot{\theta}_{a2} \end{bmatrix} \quad 7.$$

With these two Jacobian we can analyze the singularities between the actuators and the rotation of the passive joints:

$$\det(J_r) = \cos(\theta_{p1})\sin(\theta_{p2}) - \sin(\theta_{p1})\cos(\theta_{p2}) = \sin(\theta_{p1} - \theta_{p2}) \quad 8.$$

$$\exists J_r^{-1} \leftrightarrow \theta_{p1} - \theta_{p2} \neq k\pi, k \in \mathbb{N} \quad 9.$$

This condition is asking for the 2nd and 3rd links to not be parallel.

If J_r is invertible then we have:

$$\omega_r = J_r^{-1} J_q q = \Lambda_q^r q \quad 10.$$

$$\begin{aligned} \Lambda_q^r &= \frac{1}{\sin(\theta_{p1} - \theta_{p2})} \begin{bmatrix} \cos(\theta_{p2}) & -\sin(\theta_{p2}) \\ \cos(\theta_{p1}) & -\sin(\theta_{p1}) \end{bmatrix} \begin{bmatrix} \sin(\theta_{a1}) & -\sin(\theta_{a2}) \\ \cos(\theta_{a1}) & -\cos(\theta_{a2}) \end{bmatrix} = \\ &= \frac{1}{\sin(\theta_{p1} - \theta_{p2})} \begin{bmatrix} \text{left} & \text{as an} \\ \text{exercise to} & \text{the reader} \end{bmatrix} = \\ &= \frac{1}{\sin(\theta_{p1} - \theta_{p2})} \begin{bmatrix} \sin(\theta_{a1} - \theta_{p2}) & \sin(\theta_{p2} - \theta_{a2}) \\ \sin(\theta_{a1} - \theta_{p1}) & \sin(\theta_{p1} - \theta_{a2}) \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \end{aligned} \quad 11.$$

$$\begin{bmatrix} \dot{\theta}_{p1} \\ \dot{\theta}_{p2} \end{bmatrix} = \frac{1}{\sin(\theta_{p1} - \theta_{p2})} \begin{bmatrix} \sin(\theta_{a1} - \theta_{p2}) & \sin(\theta_{p2} - \theta_{a2}) \\ \sin(\theta_{a1} - \theta_{p1}) & \sin(\theta_{p1} - \theta_{a2}) \end{bmatrix} \begin{bmatrix} \dot{\theta}_{a1} \\ \dot{\theta}_{a2} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{a1} \\ \dot{\theta}_{a2} \end{bmatrix} \quad 12.$$

3) Calculations

Some notation first:

x_i, y_i are the cartesian positions of the COG of the i-th link

θ_i is the angle of the i-th link.

$\lambda_{i,j}$ is the element of the i-th row, j-th column of the mass jacobian.

3.1) Link 1

$$\begin{cases} x_1 = \frac{L}{2}\cos(\theta_{a1}) \\ y_1 = \frac{L}{2}\sin(\theta_{a1}) \\ \theta_1 = \theta_{a1} \end{cases} \quad 13.$$

$$\begin{aligned} \lambda_{1,1} &= \frac{\partial x_1}{\partial \theta_{a1}} = -\frac{L}{2}\sin(\theta_{a1}) \\ \lambda_{2,1} &= \frac{\partial y_1}{\partial \theta_{a1}} = \frac{L}{2}\cos(\theta_{a1}) \\ \lambda_{3,1} &= \frac{\partial \theta_1}{\partial \theta_{a1}} = 1 \end{aligned} \tag{14.}$$

$$\lambda_{1,2} = \lambda_{2,2} = \lambda_{2,3} = 0 \tag{15.}$$

3.2) Link 2

Kinematic chain from A1 to 2nd link:

$$\begin{cases} x_2\left(\theta_{a1}, \theta_{p1}\right) = L\cos(\theta_{a1}) + \frac{L}{2}\cos\left(\theta_{p1}\right) \\ y_2\left(\theta_{a1}, \theta_{p1}\right) = L\sin(\theta_{a1}) + \frac{L}{2}\sin\left(\theta_{p1}\right) \\ \theta_2\left(\theta_{a1}, \theta_{p1}\right) = \theta_{p1} \end{cases} \tag{16.}$$

$$\begin{aligned} \lambda_{4,1} &= \frac{\partial x_2}{\partial \theta_{a1}} = -L\sin(\theta_{a1}) + \frac{L}{2}\frac{\partial \cos\left(\theta_{p1}\right)}{\partial \theta_{a1}} = \\ &= -L\sin(\theta_{a1}) + \frac{L}{2}\frac{\partial \cos\left(\theta_{p1}\right)}{\partial \theta_{p1}}\frac{\partial \theta_{p1}}{\partial \theta_{a1}} = -L\sin(\theta_{a1}) - \frac{L}{2}\sin\left(\theta_{p1}\right)g_{11} \\ \lambda_{5,1} &= \frac{\partial y_2}{\partial \theta_{a1}} = L\cos(\theta_{a1}) + \frac{L}{2}\cos\left(\theta_{p1}\right)g_{11} \\ \lambda_{6,1} &= \frac{\partial \theta_2}{\partial \theta_{a1}} = g_{11} \end{aligned} \tag{17.}$$

Now we follow the kinematic chain from the opposite direction (A2 to 2nd link)

$$\begin{cases} x_2\left(\theta_{a2}, \theta_{p2}, \theta_{p1}\right) = 2L + L\cos(\theta_{a2}) + L\cos\left(\theta_{p2}\right) - \frac{L}{2}\cos\left(\theta_{p1}\right) \\ y_2\left(\theta_{a2}, \theta_{p2}, \theta_{p1}\right) = L\sin(\theta_{a2}) + L\sin\left(\theta_{p2}\right) - \frac{L}{2}\sin\left(\theta_{p1}\right) \\ \theta_2\left(\theta_{a2}, \theta_{p2}, \theta_{p1}\right) = \theta_{p1} \end{cases} \tag{18.}$$

$$\begin{aligned} \lambda_{4,2} &= \frac{\partial x_2}{\partial \theta_{a2}} = -L\sin(\theta_{a2}) - L\sin\left(\theta_{p2}\right)g_{22} + \frac{L}{2}\sin\left(\theta_{p1}\right)g_{12} \\ \lambda_{5,2} &= \frac{\partial y_2}{\partial \theta_{a2}} = L\cos(\theta_{a2}) + L\cos\left(\theta_{p2}\right)g_{22} - \frac{L}{2}\cos\left(\theta_{p1}\right)g_{12} \\ \lambda_{6,2} &= \frac{\partial \theta_2}{\partial \theta_{a2}} = g_{12} \end{aligned} \tag{19.}$$

3.3) Link 3

$$\begin{cases} x_3\left(\theta_{a1}, \theta_{p1}, \theta_{p2}\right) = L\cos(\theta_{a1}) + L\cos\left(\theta_{p1}\right) - \frac{L}{2}\cos\left(\theta_{p2}\right) \\ y_3\left(\theta_{a1}, \theta_{p1}, \theta_{p2}\right) = L\sin(\theta_{a1}) + L\sin\left(\theta_{p1}\right) - \frac{L}{2}\sin\left(\theta_{p2}\right) \\ \theta_3\left(\theta_{a1}, \theta_{p1}, \theta_{p2}\right) = \theta_{p2} \end{cases} \tag{20.}$$

$$\begin{aligned}\lambda_{7,1} &= \frac{\partial x_3}{\partial \theta_{a1}} = -L\sin(\theta_{a1}) - L\sin(\theta_{p1})g_{11} + \frac{L}{2}\sin(\theta_{p2})g_{21}\\ \lambda_{8,1} &= \frac{\partial y_3}{\partial \theta_{a1}} = L\cos(\theta_{a1}) + L\cos(\theta_{p1})g_{11} - \frac{L}{2}\cos(\theta_{p2})g_{21}\\ \lambda_{9,1} &= \frac{\partial \theta_3}{\partial \theta_{a1}} = g_{21}\end{aligned} \quad 21.$$

$$\begin{cases} x_3(\theta_{a2},\theta_{p2}) = 2L + L\cos(\theta_{a2}) + \frac{L}{2}\cos(\theta_{p2})\\ y_3(\theta_{a2},\theta_{p2}) = L\sin(\theta_{a2}) + \frac{L}{2}\sin(\theta_{p2})\\ \theta_3(\theta_{a2},\theta_{p2}) = \theta_{p2}\end{cases} \quad 22.$$

$$\begin{aligned}\lambda_{7,2} &= \frac{\partial x_3}{\partial \theta_{a2}} = -L\sin(\theta_{a2}) - \frac{L}{2}\sin(\theta_{p2})g_{22}\\ \lambda_{8,2} &= \frac{\partial y_3}{\partial \theta_{a2}} = L\cos(\theta_{a2}) + \frac{L}{2}\cos(\theta_{p2})g_{22}\\ \lambda_{9,2} &= \frac{\partial \theta_3}{\partial \theta_{a2}} = g_{22}\end{aligned} \quad 23.$$

3.4) Link 4

$$\begin{cases} x_4 = 2L + \frac{L}{2}\cos(\theta_{a2})\\ y_4 = \frac{L}{2}\sin(\theta_{a2})\\ \theta_4 = \theta_{a2}\end{cases} \quad 24.$$

$$\lambda_{10,1} = \lambda_{11,1} = \lambda_{12,1} = 0 \quad 25.$$

$$\begin{aligned}\lambda_{10,2} &= \frac{\partial x_4}{\partial \theta_{a2}} = -\frac{L}{2}\sin(\theta_{a2})\\ \lambda_{11,2} &= \frac{\partial y_4}{\partial \theta_{a2}} = \frac{L}{2}\cos(\theta_{a2})\\ \lambda_{12,2} &= \frac{\partial \theta_4}{\partial \theta_{a2}} = 1\end{aligned} \quad 26.$$

4) Implementation

For a given (x,y) workspace coordinate, we use the inverse kinematics to calculate $\theta_{\mathrm{ai}}, \theta_{\mathrm{pi}}$.

Then we fill the mass jacobian according to the previously specified laws.

$$[\Lambda_m] = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \\ \lambda_{3,1} & \lambda_{3,2} \\ \lambda_{4,1} & \lambda_{4,2} \\ \lambda_{5,1} & \lambda_{5,2} \\ \lambda_{6,1} & \lambda_{6,2} \\ \lambda_{7,1} & \lambda_{7,2} \\ \lambda_{8,1} & \lambda_{8,2} \\ \lambda_{9,1} & \lambda_{9,2} \\ \lambda_{10,1} & \lambda_{10,2} \\ \lambda_{11,1} & \lambda_{11,2} \\ \lambda_{12,1} & \lambda_{12,2} \end{bmatrix} \quad 27.$$

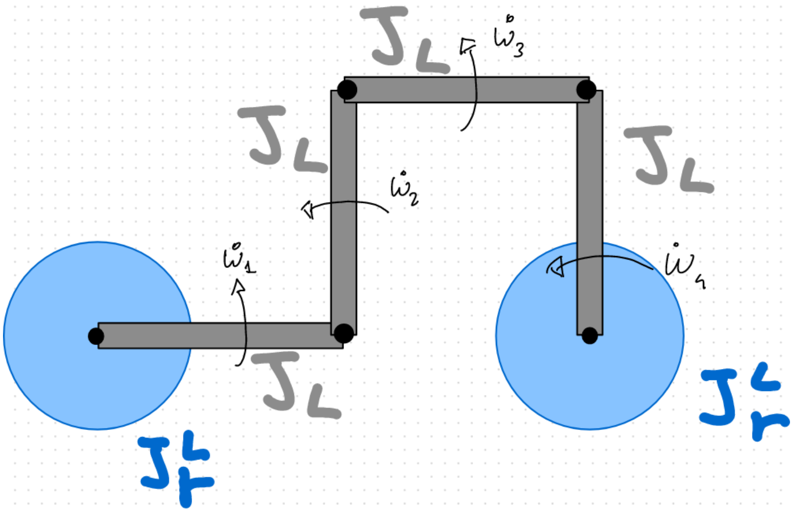
We calculate the generalized matrix of inertia with

$$[m_0] = [\Lambda_m]^T [m_{\text{ph}}] [\Lambda_m] \quad 28.$$

where $[m_{\text{ph}}] = \text{diag}\ [m, m, J_l + r^2 J_r, m, m, J_l, m, m, J_l, m, m, J_l + r^2 J_r]$

4.0.a) Why is the inertia of the first and last links like that?

Assuming to have two actuator on the 1st and 2nd load shafts, the equivalent mechanical system is this:



J_l is the moment of inertia of the links (provided by the manufacturer)

J_r is the total moment of inertia of the motor + gears assembly seen by the motor side (provided by the manufacturer), we can move J_r to the load side by multiplying by r^2 (r = gear ratio = 70): $J_r^L = J_r r^2$

We can simply sum them and place the result in the inertia of the first and last links.

4.1) Symbolic analysis

J_a = Inertia of active links (I & IV), J_p = Inertia of passive links (II & III)

$$[m_0] = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \\ \lambda_{3,1} & \lambda_{3,2} \\ \lambda_{4,1} & \lambda_{4,2} \\ \lambda_{5,1} & \lambda_{5,2} \\ \lambda_{6,1} & \lambda_{6,2} \\ \lambda_{7,1} & \lambda_{7,2} \\ \lambda_{8,1} & \lambda_{8,2} \\ \lambda_{9,1} & \lambda_{9,2} \\ \lambda_{10,1} & \lambda_{10,2} \\ \lambda_{11,1} & \lambda_{11,2} \\ \lambda_{12,1} & \lambda_{12,2} \end{bmatrix}^T \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_a \end{bmatrix} \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \\ \lambda_{3,1} & \lambda_{3,2} \\ \lambda_{4,1} & \lambda_{4,2} \\ \lambda_{5,1} & \lambda_{5,2} \\ \lambda_{6,1} & \lambda_{6,2} \\ \lambda_{7,1} & \lambda_{7,2} \\ \lambda_{8,1} & \lambda_{8,2} \\ \lambda_{9,1} & \lambda_{9,2} \\ \lambda_{10,1} & \lambda_{10,2} \\ \lambda_{11,1} & \lambda_{11,2} \\ \lambda_{12,1} & \lambda_{12,2} \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda_{1,1} & \lambda_{2,1} & \lambda_{3,1} & \lambda_{4,1} & \lambda_{5,1} & \lambda_{6,1} & \lambda_{7,1} & \lambda_{8,1} & \lambda_{8,1} & \lambda_{10,1} & \lambda_{11,1} & \lambda_{12,1} \\ \lambda_{1,2} & \lambda_{2,2} & \lambda_{3,2} & \lambda_{4,2} & \lambda_{5,2} & \lambda_{6,2} & \lambda_{7,2} & \lambda_{8,2} & \lambda_{8,2} & \lambda_{10,2} & \lambda_{11,2} & \lambda_{12,2} \end{bmatrix} \begin{bmatrix} \lambda_{1,1}m & \lambda_{1,2}m \\ \lambda_{2,1}m & \lambda_{2,2}m \\ \lambda_{3,1}J_a & \lambda_{3,2}J_a \\ \lambda_{4,1}m & \lambda_{4,2}m \\ \lambda_{5,1}m & \lambda_{5,2}m \\ \lambda_{6,1}J_p & \lambda_{6,2}J_p \\ \lambda_{7,1}m & \lambda_{7,2}m \\ \lambda_{8,1}m & \lambda_{8,2}m \\ \lambda_{9,1}J_p & \lambda_{9,2}J_p \\ \lambda_{10,1}m & \lambda_{10,2}m \\ \lambda_{11,1}m & \lambda_{11,2}m \\ \lambda_{12,1}J_a & \lambda_{12,2}J_a \end{bmatrix} = \quad 29.$$

$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

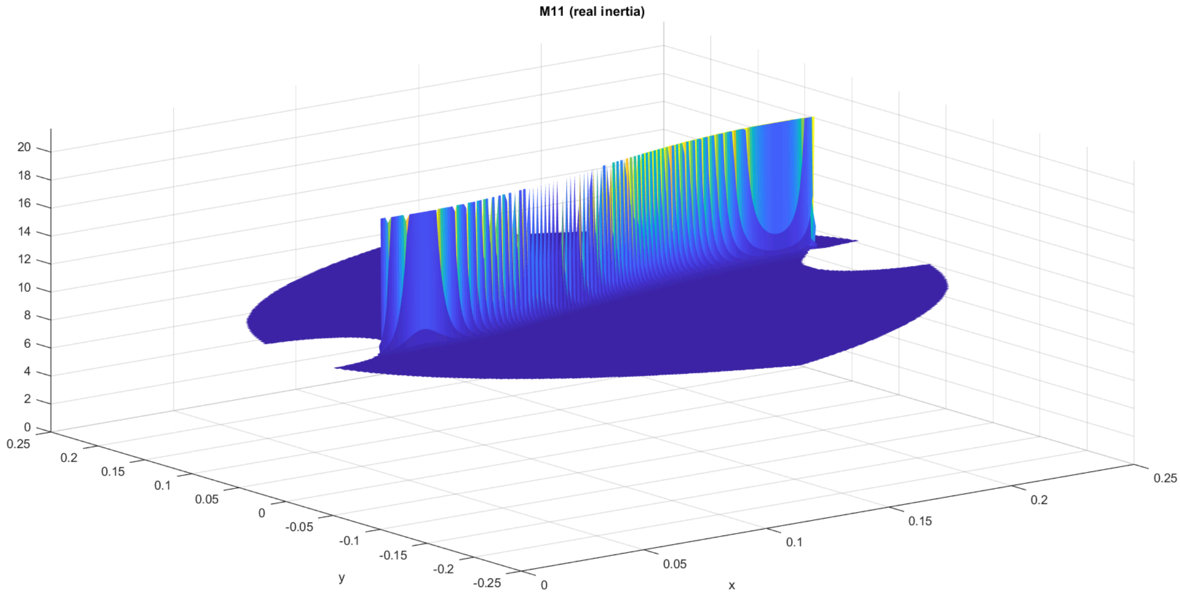
$$m_{11} = m\lambda_{1,1}^2 + m\lambda_{2,1}^2 + J_a\lambda_{3,1}^2 + m\lambda_{4,1}^2 + m\lambda_{5,1}^2 + J_p\lambda_{6,1}^2 + m\lambda_{7,1}^2 + m\lambda_{8,1}^2 + J_p\lambda_{9,1}^2 +$$
$$m\lambda_{10,1}^2 + m\lambda_{11,1}^2 + J_a\lambda_{12,1}^2 = \quad 30.$$
$$= m\lambda_{1,1}^2 + m\lambda_{2,1}^2 + J_a\lambda_{3,1}^2 + m\lambda_{4,1}^2 + m\lambda_{5,1}^2 + J_p\lambda_{6,1}^2 + m\lambda_{7,1}^2 + m\lambda_{8,1}^2 + J_p\lambda_{9,1}^2$$

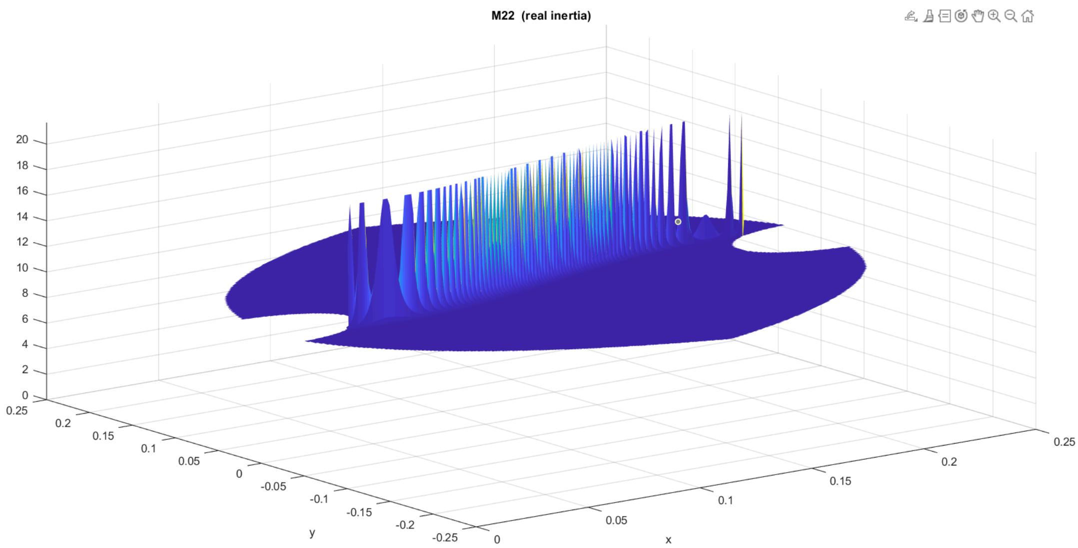
$$m_{22} = m\lambda_{1,2}^2 + m\lambda_{2,2}^2 + J_a\lambda_{3,2}^2 + m\lambda_{4,2}^2 + m\lambda_{5,2}^2 + J_p\lambda_{6,2}^2 + m\lambda_{7,2}^2 + m\lambda_{8,2}^2 + J_p\lambda_{9,2}^2$$
$$+m\lambda_{10,2}^2 + m\lambda_{11,2}^2 + J_a\lambda_{12,2}^2 = \quad 31.$$
$$= m\lambda_{4,2}^2 + m\lambda_{5,2}^2 + J_p\lambda_{6,2}^2 + m\lambda_{7,2}^2 + m\lambda_{8,2}^2 + J_p\lambda_{9,2}^2 + m\lambda_{10,2}^2 + m\lambda_{11,2}^2 + J_a\lambda_{12,2}^2$$

$$m_{12} = m_{21} = m\lambda_{1,2}\lambda_{2,1} + m\lambda_{2,2}\lambda_{2,2} + J_a\lambda_{3,2}\lambda_{2,3} + m\lambda_{4,2}\lambda_{2,4} + m\lambda_{5,2}\lambda_{2,5} + J_p\lambda_{6,2}\lambda_{2,6}$$
$$+m\lambda_{7,2}\lambda_{2,7} + m\lambda_{8,2}\lambda_{2,8} + J_p\lambda_{9,2}\lambda_{2,9} + m\lambda_{10,2}\lambda_{2,10} + m\lambda_{11,2}\lambda_{2,11} + J_a\lambda_{12,2}\lambda_{2,12} = \quad 32.$$
$$= m\lambda_{4,2}\lambda_{2,4} + m\lambda_{5,2}\lambda_{2,5} + J_p\lambda_{6,2}\lambda_{2,6} + m\lambda_{7,2}\lambda_{2,7} + m\lambda_{8,2}\lambda_{2,8} + J_p\lambda_{9,2}\lambda_{2,9}$$

5) Results

Sampling the workspace the generalized matrix of inertia has the following values:


M11 (real inertia)
x
y


M22 (real inertia)
x
y

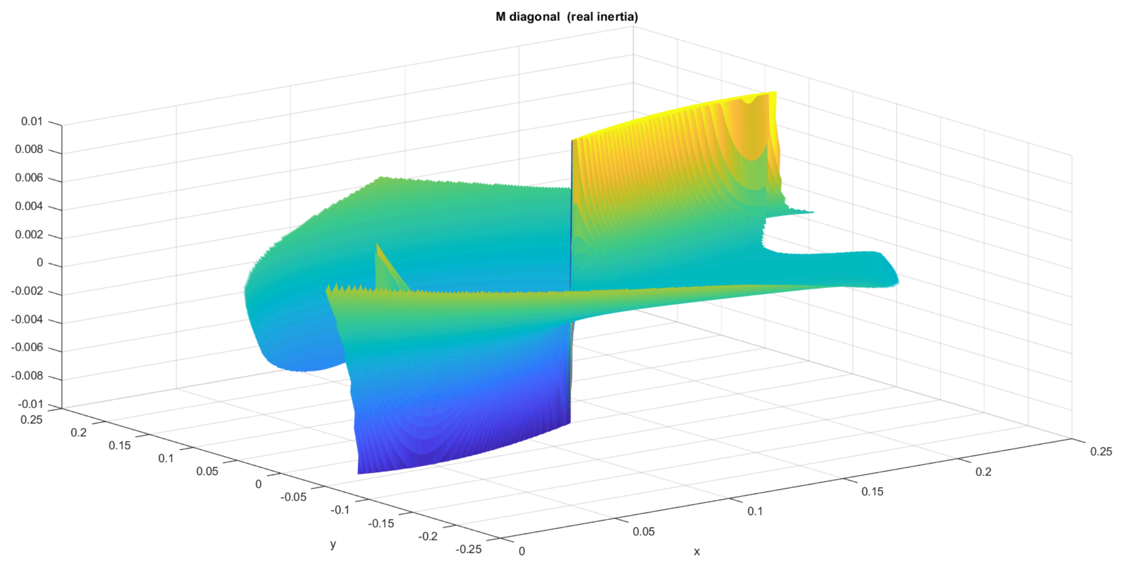
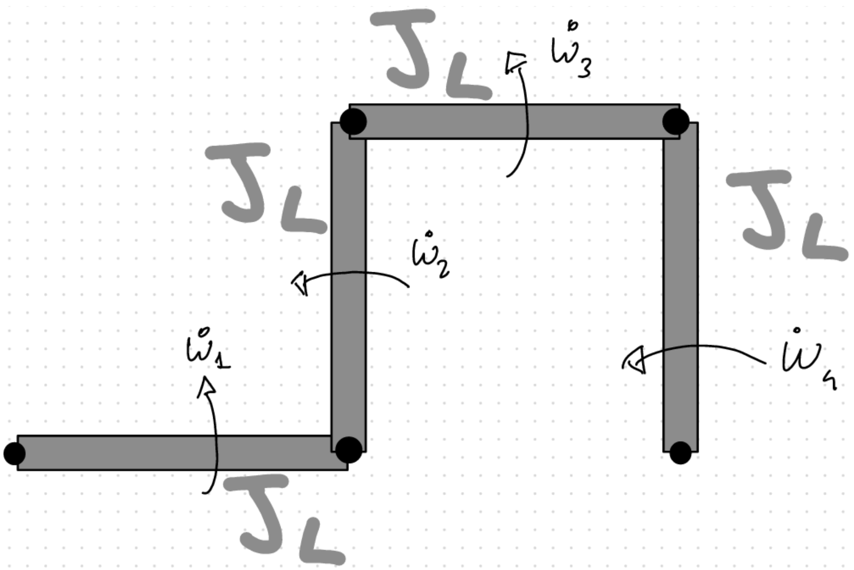


Figure 1: (N.b: ignore the image title, this technically "off-diagonal")

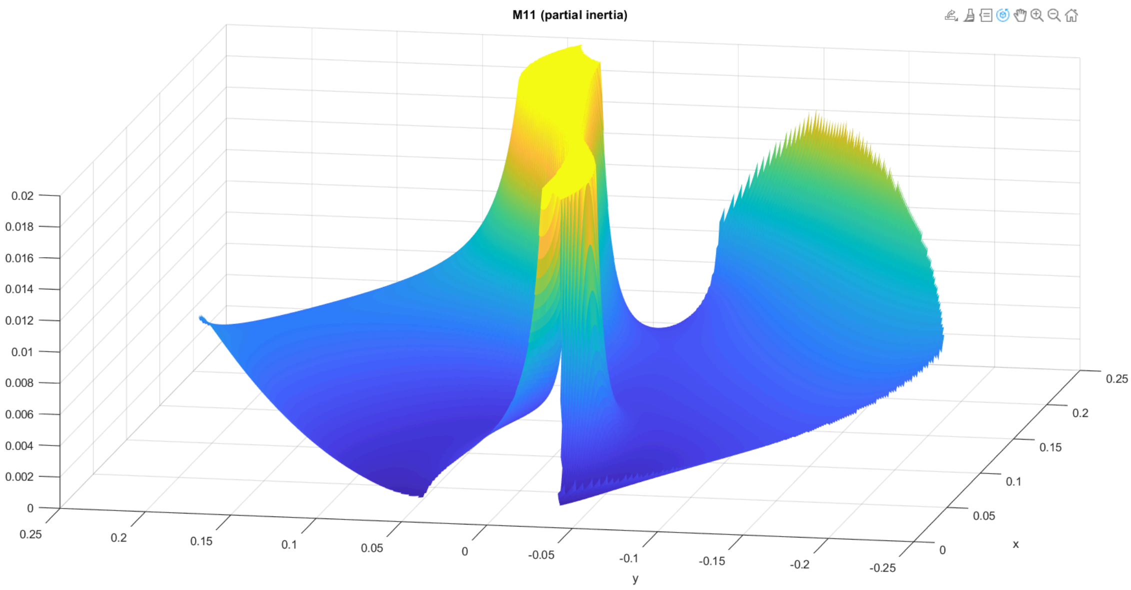
The 2 diagonal terms have a constant ≈ 10 kg m^2 throughout the workspace. With some spikes at infinity in the singularities. (Unstoppable force vs unmovable object = ∞). (For the sake of visualization the values have been capped at 20.)

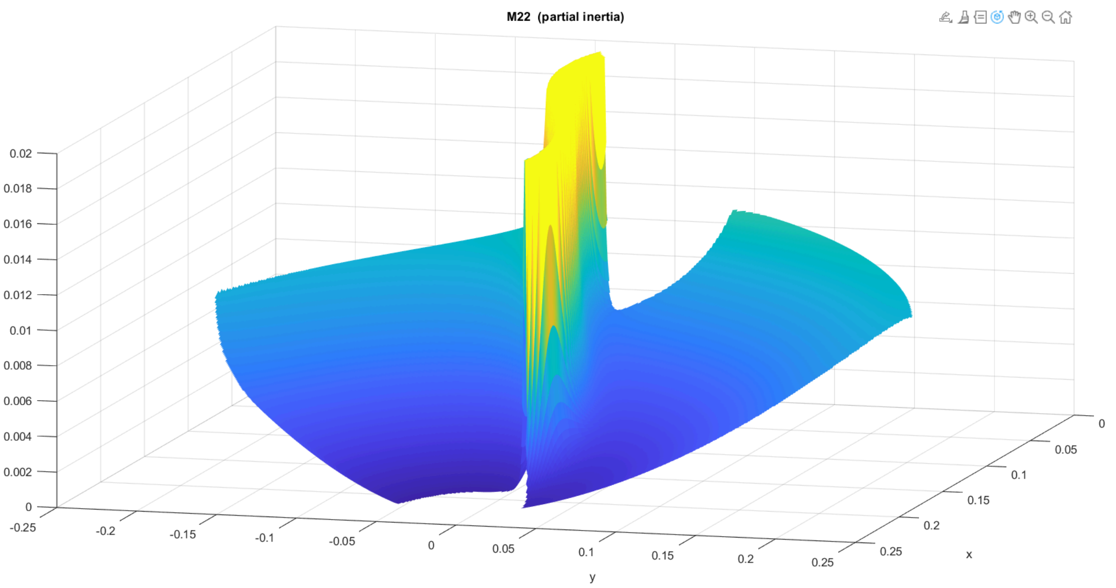
5.1) What if we forgot about the inertia before the gearbox?

For the sake of the argument let's assume we forgot to account for the inertia of the motor assembly:



We can sample again the system and see how it changes:





(Capped at 0.02)

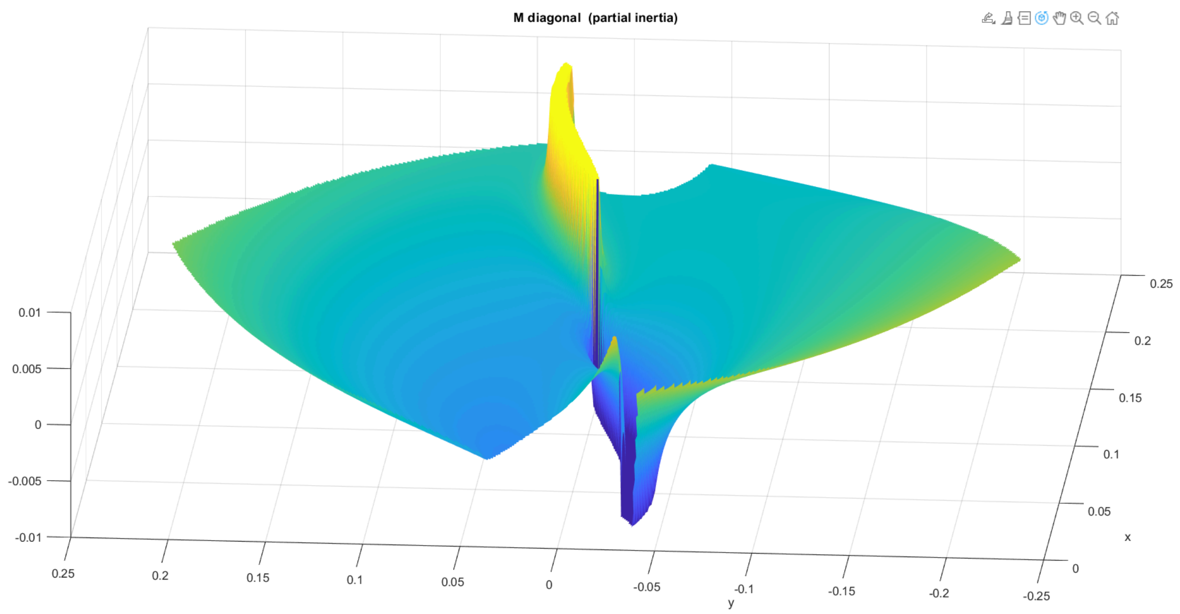


Figure 2: (N.b: ignore the image title, this technically "off-diagonal")

We can see how the inertia of the diagonal terms is much smaller and less "stable" This demonstrates the "decoupling effect of high gear ratios in robotics" which means that we can design a controller assuming a "constant" inertia.

By comparing the two off-diagonal graphs we can see that they are basically the same. This is consistent with Equation 32 where J_a doesn't appear.

6) Takeaway from this

- The inertia seen by the motor is constant.
- Simply use the value seen at the rest point (10.2338 & 10.2346) and translate it to the motor shaft.

$$J_1 = \frac{10.2338}{70^2} = 0.00208 \text{ kg } m^2 \tag{33.}$$

$$J_2 = \frac{10.2346}{70^2} = 0.00208 \text{ kg } m^2 \tag{34.}$$

- Hopefully this is consistent with time constants and stuff somehow.