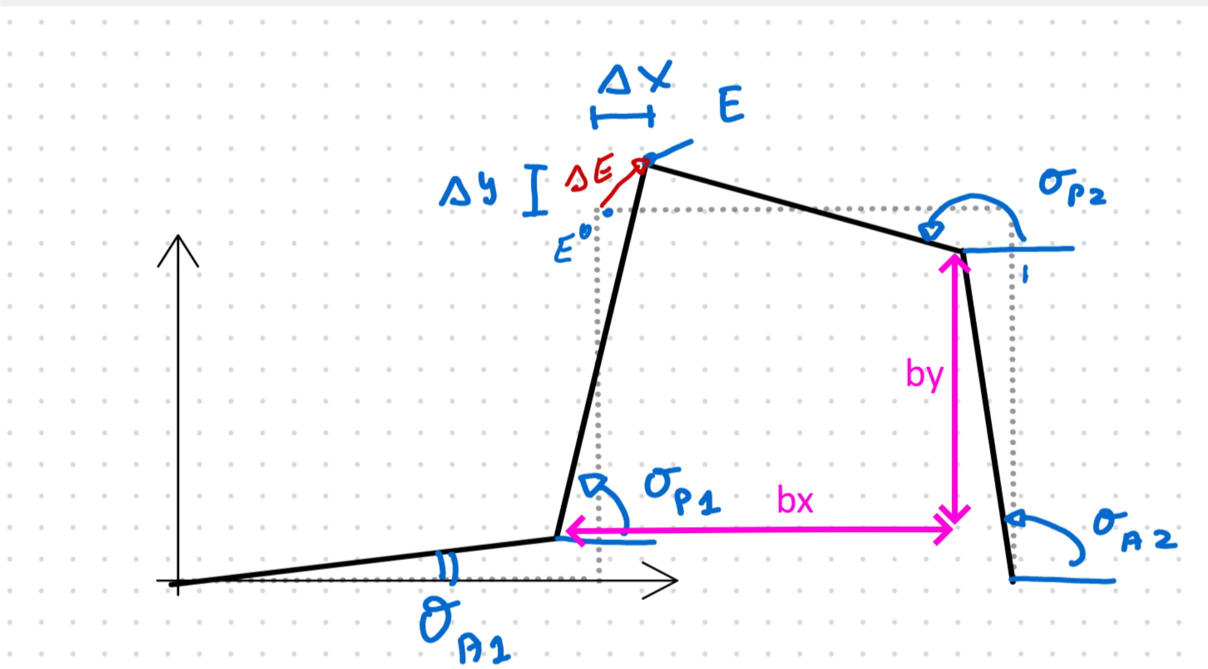
Singularity Analysis

1) Kinematic Equations

(Angle conventions taken from "A Method for Optimal Kinematic Design of Five-bar Planar Parallel Manipulators")



The kinematic chain in polar form is written as:

$$Le^{j\theta_{a1}} + Le^{j\theta_{p1}} = 2L + Le^{j\theta_{a2}} + Le^{j\theta_{p2}} \quad (1.)$$

Deriving in time and dividing by L:

$$\dot{\theta}_{a1} e^{j(\theta_{a1}+\frac{\pi}{2})} + \dot{\theta}_{p1} e^{j(\theta_{p1}+\frac{\pi}{2})} = \dot{\theta}_{a2} e^{j(\theta_{a2}+\frac{\pi}{2})} + \dot{\theta}_{p2} e^{j(\theta_{p2}+\frac{\pi}{2})} \quad (2.)$$

In Cartesian form:

$$\begin{cases} \dot{\theta}_{a1}\sin(\theta_{a1}) + \dot{\theta}_{p1}\sin(\theta_{p1}) = \dot{\theta}_{a2}\sin(\theta_{a2}) + \dot{\theta}_{p2}\sin(\theta_{p2}) \\ \dot{\theta}_{a1}\cos(\theta_{a1}) + \dot{\theta}_{p1}\cos(\theta_{p1}) = \dot{\theta}_{a2}\cos(\theta_{a2}) + \dot{\theta}_{p2}\cos(\theta_{p2}) \end{cases} \quad (3.)$$

2) Rotational Singularities

Let's define

$$\omega_r = \begin{bmatrix} \dot{\theta}_{p1} \\ \dot{\theta}_{p2} \end{bmatrix}, q = \begin{bmatrix} \dot{\theta}_{a1} \\ \dot{\theta}_{a2} \end{bmatrix} \quad (4.)$$

We want to find a form like

$$J_r \omega_r = J_q q \quad (5.)$$

(taken from "Inverse kinematics and singularity analysis of a redundant parallel robot ")

We can rearrange to have all the passive and active joint angles on one side:

$$\begin{cases} \dot{\theta}_{a1}\sin(\theta_{a1}) - \dot{\theta}_{p2}\sin(\theta_{p2}) = -\dot{\theta}_{p1}\sin(\theta_{p1}) + \dot{\theta}_{a2}\sin(\theta_{a2}) \\ \dot{\theta}_{a1}\cos(\theta_{a1}) - \dot{\theta}_{p2}\cos(\theta_{p2}) = -\dot{\theta}_{p1}\cos(\theta_{p1}) + \dot{\theta}_{a2}\cos(\theta_{a2}) \end{cases} \quad 6.$$

Convert it in matrix form

$$\begin{bmatrix} \sin(\theta_{p1}) & -\sin(\theta_{p2}) \\ \cos(\theta_{p1}) & -\cos(\theta_{p2}) \end{bmatrix} \begin{bmatrix} \dot{\theta}_{p1} \\ \dot{\theta}_{p2} \end{bmatrix} = \begin{bmatrix} -\sin(\theta_{a1}) & \sin(\theta_{a2}) \\ -\cos(\theta_{a1}) & \cos(\theta_{a2}) \end{bmatrix} \begin{bmatrix} \dot{\theta}_{a1} \\ \dot{\theta}_{a2} \end{bmatrix} \quad 7.$$

With these two Jacobian we can analyze the singularities between the actuators and the rotation of the passive joints:

$$\det(J_r) = \cos(\theta_{p1})\sin(\theta_{p2}) - \sin(\theta_{p1})\cos(\theta_{p2}) = \sin(\theta_{p1} - \theta_{p2}) \quad 8.$$

$$\exists J_r^{-1} \leftrightarrow \theta_{p1} - \theta_{p2} \neq k\pi, k \in \mathbb{N} \quad 9.$$

This condition is asking for the 2nd and 3rd links to not be parallel.

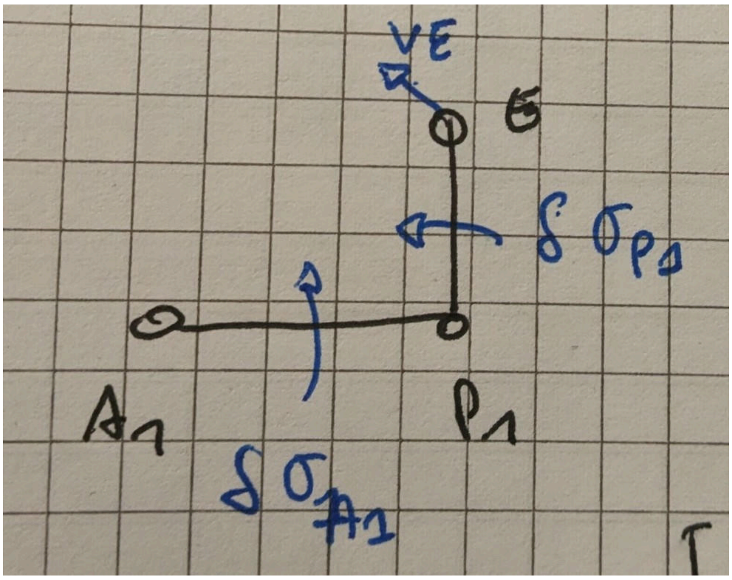
If J_r is invertible then we have:

$$\omega_r = J_r^{-1} J_q q = \Lambda_q^r q \quad 10.$$

$$\begin{aligned} \Lambda_q^r &= \frac{1}{\sin(\theta_{p1} - \theta_{p2})} \begin{bmatrix} -\cos(\theta_{p2}) & \sin(\theta_{p2}) \\ -\cos(\theta_{p1}) & \sin(\theta_{p1}) \end{bmatrix} \begin{bmatrix} -\sin(\theta_{a1}) & \sin(\theta_{a2}) \\ -\cos(\theta_{a1}) & \cos(\theta_{a2}) \end{bmatrix} = \\ &= \frac{1}{\sin(\theta_{p1} - \theta_{p2})} \begin{bmatrix} \text{left} & \text{as an} \\ \text{exercise to} & \text{the reader} \end{bmatrix} = \\ &= \frac{1}{\sin(\theta_{p1} - \theta_{p2})} \begin{bmatrix} \sin(\theta_{a1} - \theta_{p2}) & \sin(\theta_{p2} - \theta_{a2}) \\ \sin(\theta_{a1} - \theta_{p1}) & \sin(\theta_{p1} - \theta_{a2}) \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \end{aligned} \quad 11.$$

$$\begin{bmatrix} \dot{\theta}_{p1} \\ \dot{\theta}_{p2} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{a1} \\ \dot{\theta}_{a2} \end{bmatrix} \quad 12.$$

3) Velocity Singularities



From the linearized left side of the chain:

$$\begin{cases} \delta X_E = -L\delta\theta_{a1}\sin(\theta_{a1}) - L\delta\theta_{p1}\sin(\theta_{p1}) \\ \delta Y_E = L\delta\theta_{a1}\cos(\theta_{a1}) + L\delta\theta_{p1}\cos(\theta_{p1}) \end{cases} \quad 13.$$

In matrix form:

$$v = \begin{bmatrix} \delta X_E \\ \delta Y_E \end{bmatrix} = L \begin{bmatrix} -\sin(\theta_{a1}) & -\sin(\theta_{p1}) \\ \cos(\theta_{a1}) & \cos(\theta_{p1}) \end{bmatrix} \begin{bmatrix} \delta\theta_{a1} \\ \delta\theta_{p1} \end{bmatrix} = J_1^x \begin{bmatrix} \delta\theta_{a1} \\ \delta\theta_{p1} \end{bmatrix} \quad 14.$$

We have a relation between the angles of the left links and the velocity of the End Effector. Ideally we would like a relation between the actuators and the velocity. We can use the relations found in Equation 12 for that.

$$\begin{bmatrix} \delta\theta_{a1} \\ \delta\theta_{p1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ g_{11} & g_{12} \end{bmatrix} \begin{bmatrix} \dot\theta_{a1} \\ \dot\theta_{a2} \end{bmatrix} \quad 15.$$

$$v = J_1^x \begin{bmatrix} 1 & 0 \\ g_{11} & g_{12} \end{bmatrix} \begin{bmatrix} \dot\theta_{a1} \\ \dot\theta_{a2} \end{bmatrix} = \Lambda_q^x q \quad 16.$$

With

$$\Lambda_q^x = L \begin{bmatrix} -\sin(\theta_{a1}) & -\sin(\theta_{p1}) \\ \cos(\theta_{a1}) & \cos(\theta_{p1}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g_{11} & g_{12} \end{bmatrix} =$$
$$= L \begin{bmatrix} -\sin(\theta_{a1}) - \sin(\theta_{p1})g_{11} & -\sin(\theta_{p1})g_{11} \\ \cos(\theta_{a1}) + \cos(\theta_{p1})g_{12} & \cos(\theta_{p1})g_{12} \end{bmatrix} \quad 17.$$

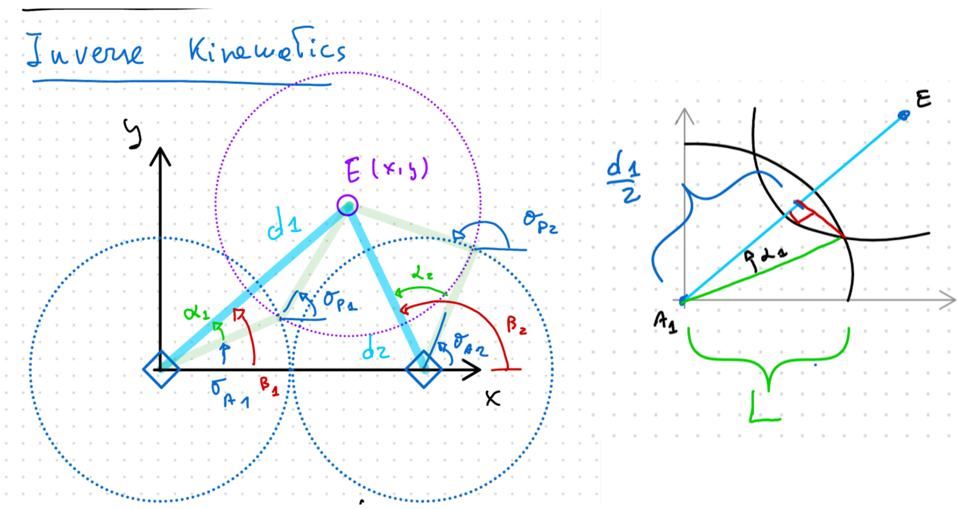
(N.b. In the MATLAB code g_{11}, g_{12} are galled g1 and g2 respectively)

In this case the singularity analysis is not as straightforward as the rotational case.

4) Inverse Kinematics

We can sample the Cartesian Workspace and see where $\det(\Lambda_q^x) = 0$.

Before doing that we need to find a function that given a Cartesian coordinate returns the joint angles.



$$\beta_1 = \tan^{-1}\left(\frac{y}{x}\right) \tag{18.}$$

$$\beta_2 = \tan^{-1}\left(\frac{y}{x-2L}\right) \tag{19.}$$

$$d_1 = \sqrt{x^2+y^2} \tag{20.}$$

$$d_2 = \sqrt{(x-2L)^2+y^2} \tag{21.}$$

$$\alpha_1 = \cos^{-1}\left(\frac{d_1}{2L}\right) \tag{22.}$$

$$\alpha_2 = \cos^{-1}\left(\frac{d_2}{2L}\right) \tag{23.}$$

$$\theta_{a1} = \beta_1 - \alpha_1 \tag{24.}$$

$$\theta_{a2} = \beta_2 - \alpha_2 \tag{25.}$$

$$\theta_{p1} = \tan^{-1}\left(\frac{y - L\sin(\theta_{a1})}{x - L\cos(\theta_{a1})}\right) \tag{26.}$$

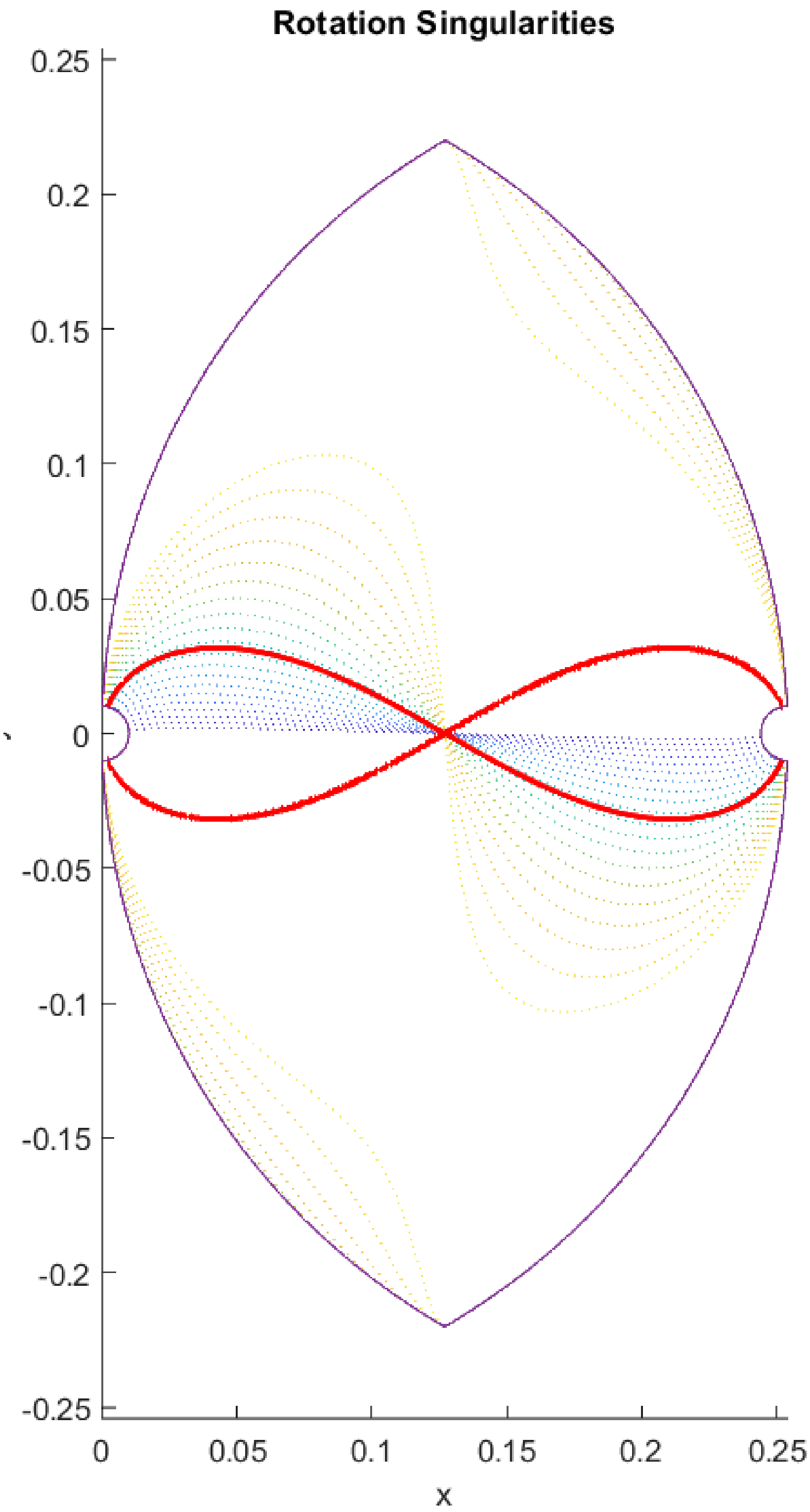
$$\theta_{p2} = \tan^{-1}\left(\frac{y - L\sin(\theta_{a2})}{x - 2L - L\cos(\theta_{a2})}\right) \tag{27.}$$

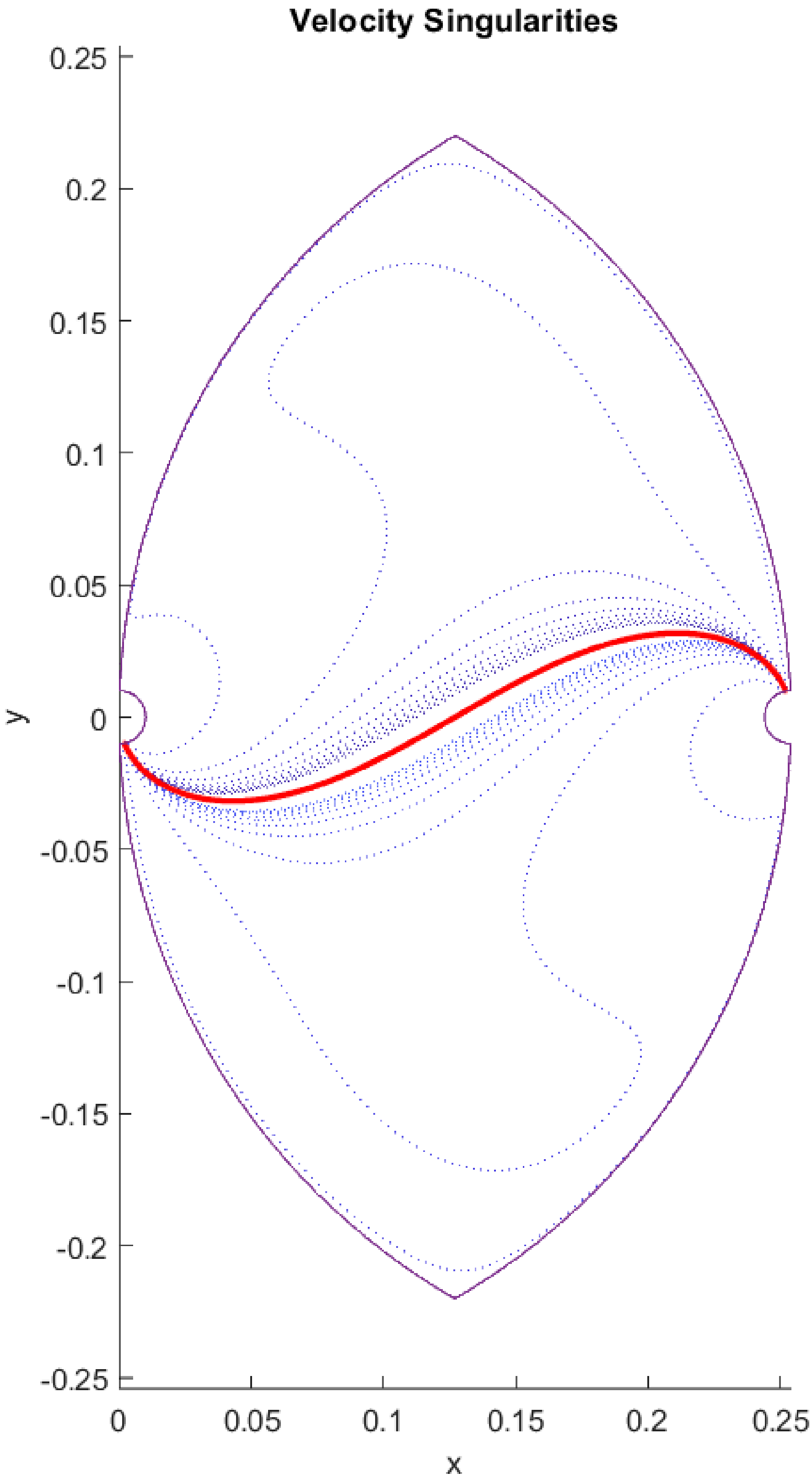
5) Computer assisted analysis

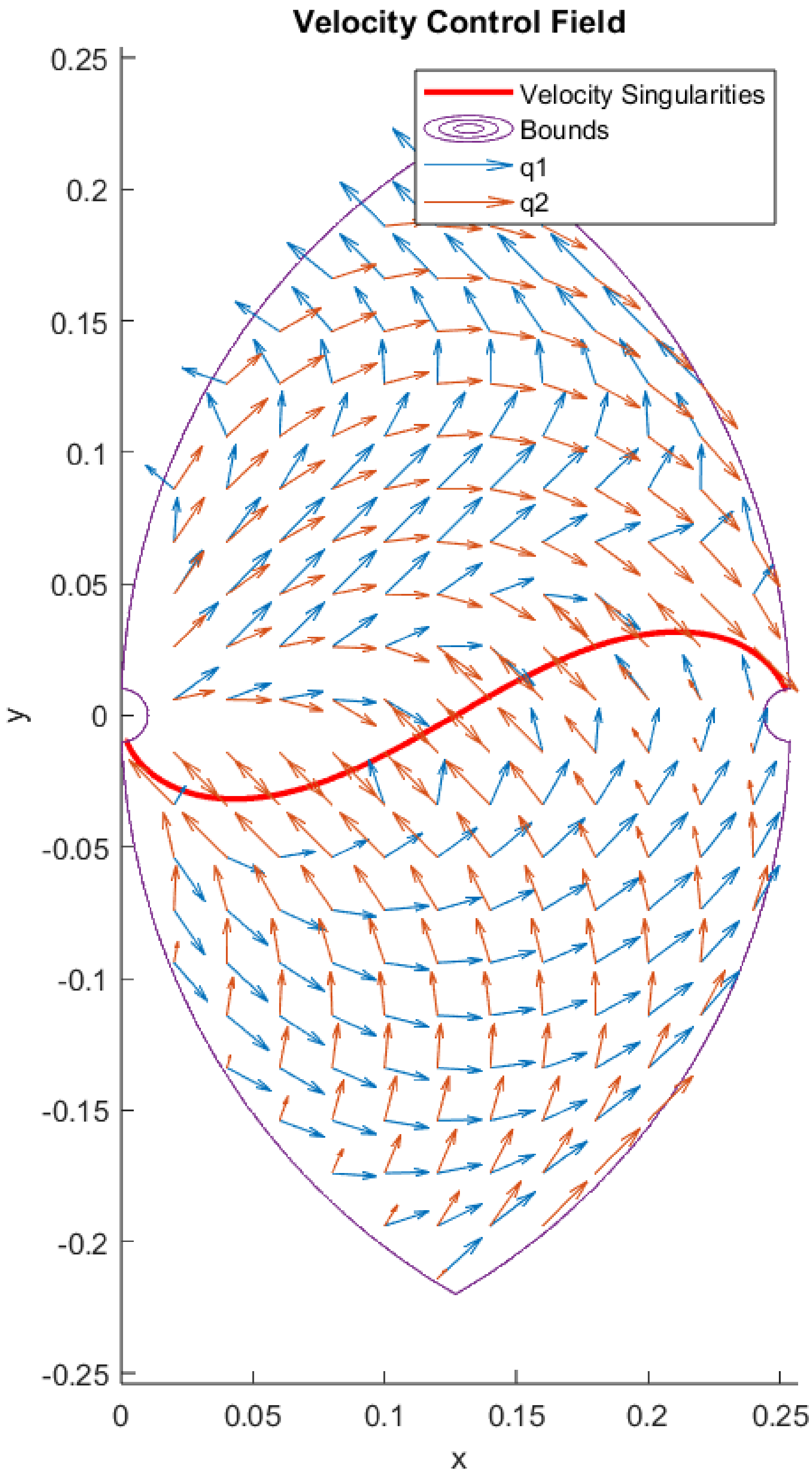
Now that we have the inverse kinematic law it's possible to analyze the properties of the robot.

```
for every sampled (x,y) in workspace
    if distance(x,y) is feasible then
        θ_ai, θ_pi ← inverseKinematics(x,y)
        calculate Λ_q^x(θ_ai θ_pi)
        calculate Λ_q^r(θ_ai θ_pi)
        return [det(Λ_q^x), Λ_q^x, det(Λ_q^r), Λ_q^r]
    else
        return error
    end
end
```

6) Results


Rotation Singularities
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
0
0.05
0.1
0.15
0.2
0.25
x


Velocity Singularities
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
y
0
0.05
0.1
0.15
0.2
0.25
x


Velocity Control Field
Velocity Singularities
Bounds
q1
q2
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
y
0
0.05
0.1
0.15
0.2
0.25
x